RESEARCH STATEMENT

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1. Introduction

My research uses *geometric analysis* to understand **noncompact** and **singular manifolds** where geometry, topology, and physics intersect. I focus on two directions:

1.1. **Higher–Curvature Gravity.** I study higher–order generalizations of Einstein metrics in dimensions ≥ 5, motivated by the *AdS/CFT correspondence* [20, 23], where bulk geometry encodes boundary conformal data. This extends the framework of *conformally compact Einstein metrics* and the singular Yamabe problem, suggesting that Lovelock equations may admit conformally compact fillings that are topologically or geometrically obstructed in the Einstein case.

My work develops boundary asymptotics for conformally compact Lovelock metrics, identifies the associated ambient obstruction tensor, and topological restrictions on admissible fillings.

1.2. **Morse–Bott Theory on Singular Spaces.** We construct Witten instanton complexes on stratified pseudomanifolds with nonisolated critical sets, developing an analytic Morse theory that unites Hodge, index–theoretic, and semiclassical techniques. This framework generalizes classical Morse inequalities and dualities to singular geometries.

Together, these projects use analytic tools, such as elliptic boundary value problems, Witten deformation, localization, and Hodge theory on singular manifolds.

Section 2 establishes boundary asymptotics and obstruction theory for conformally compact Lovelock metrics; Section 3 constructs Witten instanton complexes and Morse–Bott inequalities on singular spaces; and Section 4 outlines future directions, including pinching and rigidity results for conformally compact Lovelock metrics and cohomological formulations of spectral ξ-invariants.

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2. Boundary Asymptotics and Obstructions in Higher-Curvature Gravity

2.1. **Lovelock Equations and Conformal Geoemtry.** On an $(n+1)$-dimensional manifold, the Einstein tensor is uniquely characterized as a symmetric, divergence-free 2-tensor built from the metric and its first two derivatives, and linear in the second derivatives. Lovelock's key insight was that relaxing the assumption of linearity in the curvature tensor leads to a broader class of higher-order geometric tensors, known as the **Lovelock tensors** [19]. These are defined as linear combinations of the generalized Ricci curvatures and their traces:

$$F(\alpha) := \sum_q \alpha_q \left(\mathrm{Ric}^{(2q)} - \frac{1}{2q}\,\mathrm{scal}^{(2q)} g \right) - \lambda g = 0, \qquad 0 \leq 2q \leq n,$$

where each generalized Ricci curvature $\mathrm{Ric}^{(2q)}$ and its scalar trace $\mathrm{scal}^{(2q)}$ are given by

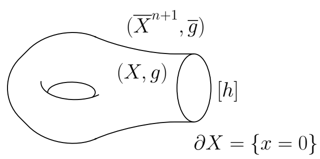
$$\big(\mathrm{Ric}^{(2q)}_g\big)_i^{\ j} = \delta^{ii_1\cdots i_{2q}}_{jj_1\cdots j_{2q}}\, R^{j_1j_2}_{i_1i_2} \cdots R^{j_{2q-1}j_{2q}}_{i_{2q-1}i_{2q}}, \qquad \mathrm{scal}^{(2q)}_g = \mathrm{tr}\big(\mathrm{Ric}^{(2q)}_g\big),$$

and $\delta^{i_1i_2\cdots i_{2q}}_{j_1j_2\cdots j_{2q}} = \det\big(\delta^{i_s}_{j_t}\big)$ denotes the generalized Kronecker delta.

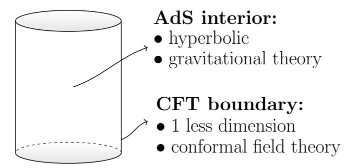
The first new case is *Gauss–Bonnet gravity*, whose action involves $\mathrm{scal}^2 - 4|\mathrm{Ric}|^2 + |\mathrm{Rm}|^2$. While topological in dimension four, it contributes nontrivially in higher dimensions.

Higher-curvature models play a central role in the **AdS/CFT correspondence** [20, 23], relating gravitational theories on asymptotically AdS spaces to conformal field theories on their boundary, and motivating the study of conformal geometry at infinity (see Figure (B) below).

A metric g on the interior of a manifold X with boundary is called **conformally compact** if there exists a boundary defining function $x \geq 0$ such that $x^2 g$ extends smoothly to $\overline{X}$. The induced boundary metric is determined only up to conformal rescaling, which gives a well-defined **conformal infinity** $[h] = [g|_{x=0}]$. A classical example is the hyperbolic ball B^{n+1}, where the hyperbolic metric compactifies to the closed ball with boundary the round sphere S^n.



(A) Conformally compact manifold



(B) AdS/CFT correspondence

Albin [2] proved that conformally compact Lovelock metrics admit a **formal Fefferman - Graham expansion**: Near the boundary

$$g = \frac{dx^2 + h_x}{x^2}, \qquad h_x = h_0 + h_2 x^2 + h_4 x^4 + \cdots + \begin{cases} h_n x^n + \cdots & n \text{ odd}, \\ h_{n,1}\, x^n \log x + h_n x^n + \cdots & n \text{ even}. \end{cases} \tag{2.1}$$

The restriction to *even powers* in the lower-order terms highlights the structural constraints imposed by the equations. For even n, the logarithmic term obstructs smoothness, and its coefficient $h_{n,1}$ defines *a new conformal invariant* called the **ambient obstruction tensor**.

2.2. **Asymptotics, Obstructions, and Topology in Higher-Order Gravity.** A key difficulty is that the Lovelock equations are nonlinear and degenerate due to diffeomorphism invariance, which prevents useful analytic control. Following Albin [2], I apply **DeTurck's gauge-fixing method** to reformulate the system in special coordinates where it becomes elliptic and linearizes to a **Laplace-type operator** with rapidly decaying lower-order terms.

To analyze its boundary behavior, I extend the inductive approach of Biquard–Herzlich [4], who established polyhomogeneity for CCE metrics by constructing a parametrix (**Green's integral operator**) and controlling the decay of successive correction terms. This iterative Green's-operator method yields precise estimates ensuring convergence of the asymptotic expansion.

Theorem A (Boundary Asymptotics)**.** *Conformally compact Lovelock metrics admit full asymptotic expansions of the form $\sum_{z,k} a_{z,k}(y)\, x^z (\log x)^k$, so in particular they realize the formal Fefferman-Graham expansion* (2.1)*.*

For even boundary dimension n, identifying the leading term of the obstruction tensor requires differentiating the Lovelock tensor, where higher-order curvature contractions create intricate combinatorial couplings. Using Labbi's *double-forms* formalism [15, 17, 16], I performed this differentiation in a fully tensorial way. Although the local formulas are complex, the *essential structure is combinatorial*: carefully tracking the coupling constants reveals cancellations that **reproduce the recurrence relation** found in the Einstein case by Graham–Hirachi [11].

Theorem B (Ambient Obstruction)**.** *Up to a constant, the leading-order term of the obstruction tensor in the generalized setting agrees with the Einstein case.*

Inspired by the singular Yamabe problem studied by Graham [10], I study rescalings $u^2 g$ with *constant generalized scalar curvature*, i.e.

$$\mathrm{scal}^{(2q)}(u^2 g) = \text{constant}.$$

Theorem C (Singular Yamabe-$(2q)$ Problem)**.** *There exists a formal solution*

$$u = x + u_2 x^2 + \cdots + u_{n+1} x^{n+1} + \mathcal{L}\, x^{n+1} \log x + u_{n+2} x^{n+2} + \cdots ,$$

where the coefficients u_k are determined recursively by the boundary geometry, and $\mathcal{L}$ is an obstruction to smoothness.

I also study the **conformally compact Lovelock filling problem**:

Problem 2.1 (Filling Problem)**.** *Given a boundary metric h, can one find a conformally compact interior metric g such that x^2g extends conformally to h and g solves the Lovelock equation?*

In the Einstein case, Gursky–Han–Stolz [12] established a **topological obstruction** via index theory and boundary geometry. In the Lovelock setting, the Fefferman–Graham expansion likewise guarantees a totally geodesic extension near the boundary, enabling a parallel gluing argument. Although Lovelock metrics need not have constant scalar curvature, the maximum principle underlying the index-theoretic approach continues to hold under a suitable lower curvature bound.

Theorem D (Topological Obstruction)**.** *Let X be a compact spin manifold with boundary. If the boundary metric h has positive Yamabe invariant (e.g. positive scalar curvature) and the Dirac operator on X has nonzero index, then no conformally compact Lovelock metric g on the interior of X can solve the Filling Problem with* $\mathrm{scal}(g) \geq -n(n+1)$.

3. Witten Instantons and Morse–Bott Inequalities

3.1. **Stratified Spaces and Witten Instanton Complexes.** Classical Morse theory relates critical points of a smooth function to the topology of a manifold. Witten [21, 22] reformulated it analytically via a deformation of the de Rham complex, whose low-energy states localize near critical points, and Bismut [5] extended this to the Morse–Bott case with critical submanifolds.

Many spaces in geometry and physics are **stratified pseudomanifolds**, where the space is decomposed into smooth strata of varying dimensions and glued together in a controlled way. Near a point p in a lower-dimensional stratum Y, a neighborhood is modeled by

$$\mathbb{R}^k \times C(Z), \qquad C(Z) = ([0,1) \times Z)/(\{0\} \times Z),$$

a cone over the link Z. Iterated cones are examples. In general, singular sets may form submanifolds of positive dimension.

The stratified space $\widehat{X}$ can be *resolved* to a manifold with corners X carrying a **wedge metric** g_w. Locally near a boundary hypersurface with base Y and fiber Z,

$$g_w \sim dx^2 + x^2 g_Z + g_Y,$$

where x is a boundary defining function ($x = 0$ on Y), g_Z a metric on the link Z, and g_Y along the stratum Y, so directions in Z scale by $1/x$ near the singular set. On X, the de Rham complex is not uniquely defined: near singular strata, one must choose which form components extend across the singularity. Such analytic domains, classified by cohomological data, are the **mezzo-perversities** studied by Albin [1] and Albin-Leichtnam-Mazzeo-Piazza [3].

Our paper [14] studies *flat stratified Morse–Bott* functions on such spaces. A *Morse function* has isolated non-degenerate critical points (e.g. the height function on a torus attains extrema along circles), while a **Morse–Bott function** h allows a smooth critical submanifold $N \subset M$ whose Hessian is non-degenerate in directions normal to N. In the stratified case, critical sets may lie within singular strata, producing more intricate local models.

3.2. **Morse-Bott inequalities.** We construct a finite-dimensional **Witten instanton complex** on X using a **deformation method**, following Witten's analytic approach.

One introduces the operator $P_\varepsilon = e^{-\varepsilon h} P e^{\varepsilon h}$, which *preserves supersymmetry* and forces eigenforms to *localize near critical sets* [21, 26, 24]. Around each critical component, a local Hilbert complex is built on a fundamental neighborhood adapted to the gradient flow. The flatness condition ensures a **separation-of-variables argument** and a **Bochner-type identity** that controls the spectrum [14]. These local harmonic representatives are then patched into global quasimodes, and the small-eigenvalue spaces remain stable under the deformed differential.

Theorem E (Witten instanton complex on stratified spaces)**.** *Let h be a flat stratified Morse-Bott function on $\widehat{X}$, and let W be a choice of mezzo-perversity. Then the Witten deformation produces a finite-dimensional* instanton complex

$$\mathcal{P}_{W,\mathrm{inst}}(X) := \big(K^{[0,c]}_{W,\varepsilon,k},\ P_\varepsilon\big),$$

generated by eigensections with sufficiently small eigenvalues. This instanton complex is quasi-isomorphic to the global twisted de Rham complex $\mathcal{P}_W(X)$.

Instead of a binary split into "small vs. large" eigenvalues, we introduce a **three-band decomposition** of the Witten Laplacian. The small eigenvalues yield a finite-dimensional instanton complex that is quasi-isomorphic to the global complex. The intermediate eigenvalue is ruled out using Bochner-type localization and fiberwise spectral gaps, while the large eigenvalue is controlled by the second order term $\epsilon^2|\nabla h|^2$ away from critical sets, providing uniform lower bounds.

From this local-to-global construction we obtain *polynomial Morse inequalities*, comparing the Morse polynomial defined from the instanton complex to the global Poincaré polynomial, up to an explicit error term. The proof interprets local contributions as polynomial traces over cohomology and then sums over all critical components.

Theorem F (Polynomial Morse inequalities)**.** *In the above setting, the* Morse polynomial *and the* Poincaré polynomial *of the global twisted de Rham complex are*

$$M(\mathcal{P}_W(X),h)(b) := \sum_{k=0}^{n} b^k \dim K^{[0,c]}_{W,\varepsilon,k}, \qquad P(\mathcal{P}_W(X))(b) := \sum_{k=0}^{n} b^k \dim H^k(\mathcal{P}_W(X)).$$

Then there exist nonnegative integers $Q_0,\dots,Q_{n-1}$ such that

$$M(\mathcal{P}_W(X),h)(b) \;=\; P(\mathcal{P}_W(X))(b) \;+\; (1+b)\sum_{k=0}^{n-1} Q_k\, b^k.$$

A further refinement appears when the mezzo-perversity is **self-dual**. In this situation, there is a natural **duality symmetry**: replacing h with $-h$ swaps the attracting and repelling directions of the gradient flow. For self-dual W, this symmetry aligns the two constructions, leading to *refined Morse inequalities*—a direct analog of classical Poincaré duality in this singular setting.

Theorem G (Refined duality for self-dual mezzo-perversities [14])**.** *If W is self-dual, then*

$$M(\mathcal{P}_W(X),h)(b) \;=\; b^n\, M(\mathcal{P}_W(X),-h)(b^{-1}).$$

4. Future Directions

4.1. **Conformally Compact Lovelock Filling Problem.** A natural next step is to develop rigidity results for conformally compact Lovelock metrics, extending the work of Chang–Yang–Zhang [6], who proved that when the conformal infinity is cylindrical, the only Poincaré–Einstein fillings are the AdS–Schwarzschild metrics on $D^2\times S^{n-1}$.

Their theory consists of two complementary components. **Qualitative rigidity** uses topological arguments—most notably the surjectivity of the induced map on fundamental groups and geodesic rigidity—to force a global splitting. **Quantitative rigidity** depends on volume comparison and curvature pinching estimates. Both rely crucially on the Einstein condition $\mathrm{Ric} = -ng$, which ensures the validity of strong maximum principles.

Problem 4.1. *If the conformal infinity is $S^1\times S^{n-1}$ and the interior satisfies* $\mathrm{Ric}\ge -ng$*, must every complete conformally compact Lovelock filling also split as $D^2\times S^{n-1}$?*

In the Lovelock setting, the field equations no longer imply $\mathrm{Ric} = -ng$. To adapt the CYZ framework, I propose imposing a lower Ricci bound $\mathrm{Ric} \geq -ng$, paralleling the scalar curvature bound $\mathrm{scal} \geq -n(n+1)$ used in Theorem D. Under such curvature assumptions, the qualitative rigidity arguments should continue to hold, and the Bishop–Gromov comparison theorem remains applicable.

The main difficulty arises in the quantitative aspect: for Einstein metrics, pinching estimates relate the sectional curvature to the hyperbolic model via refined Ricci and Weyl tensor decompositions, leading to gap theorems in Li-Qing-Shi [18]. In contrast, the Lovelock equations couple higher curvature tensors nonlinearly, so no analogous control on sectional or Weyl curvature is yet known. In this case, I expect the effective decomposition in Labbi [15] will be helpful. Developing such an estimate would be the first step toward a genuine rigidity theory for conformally compact Lovelock metrics.

Question 4.2. *Can one establish a curvature pinching estimate for complete conformally compact Lovelock metrics, that is comparable to the classical pinching results for Einstein metrics?*

A related direction concerns the existence of conformally compact *Lovelock fillings* in situations where Einstein fillings fail to exist. Unlike the Einstein equations, the Lovelock system incorporates higher-order curvature corrections, which may relax the geometric constraints underlying classical obstruction results. This leads to a natural question:

Question 4.3. *Can there exist conformally compact Lovelock metrics that realize fillings forbidden in the Einstein case?*

If such metrics exist, their curvature should approach, but not achieve, the Einstein thresholds. Identifying these borderline configurations would clarify how higher-curvature terms affect geometric rigidity and topological obstructions. It would also be illuminating to construct explicit non-fillable examples using known gravitational instantons, such as third-order Lovelock metrics of Taub–NUT type [9, 13].

4.2. **Cohomological Formulation of the ξ-Invariant.** In an ongoing project, we develop a cohomological formulation of the ξ-invariant for spaces with singularities, replacing difficult spectral computations with an algebraically computable framework.

The ξ-invariant is a refined spectral invariant of a self-adjoint Dirac-type operator on the link of a singularity, defined as half the sum of its η-invariant and the graded dimension of its kernel:

$$\xi(D) = \tfrac{1}{2}\big(\eta(D) + \dim\ker D\big).$$

It measures the local index defect arising from spectral asymmetry. In the analytic approach of Chou [7], this defect is expressed through the spectrum of the tangential Dirac operator, requiring detailed eigenvalue analysis. Our cohomological formulation provides a simpler and more explicit alternative.

Building on Zhang's equivariant η-invariant framework [25], we **decompose the Dirac operator** near a conical singularity into commuting radial and tangential components. Their graded commutativity, interpreted through supersymmetry, yields a natural pairing of eigensections and explains the cancellations underlying spectral invariants. This leads to a unified analytic–algebraic interpretation of the index defect:

Claim 4.4. *The spectral ξ-invariant defined via eigenvalue regularization coincides with a supertrace over local cohomology groups.*

Our current results concern cones endowed with almost complex structures and twisted Dolbeault–Dirac operators, where this equivalence can be established directly. Explicit computations are obtained for spheres, orbifolds, and algebraic varieties studied by Degeratu [8] and Jayasinghe [14]. We also plan to extend this framework to stratified pseudomanifolds.

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