Sep 9 initial data (M^n, g_0) compact.

$$\begin{cases} \partial_t g(t) = -2\,Ric(g(t)) \\ g(0) = g_0 \end{cases}$$

Ricci flow

Hamilton ~~...~~ $g(t)$ exists and unique on $[0,T)$

and $\sup_M |Rm|(t) \xrightarrow{t \nearrow T} \infty$ $T < \infty$.

parabolic blow up. $\longrightarrow$ singularity models.

Take $\{(x_i, t_i)\}$ $t_i \to T$

$$\text{"} |Rm(x_i, t_i)| = \sup_{M \times [0, t_i]} |Rm| \text{"} = \lambda_i$$

Consider $g_i(t) = \lambda_i\, g_i(t_i + \lambda_i^{-1} t)$, $t \in [-\lambda_i t_i, \lambda_i (T - t_i)]$

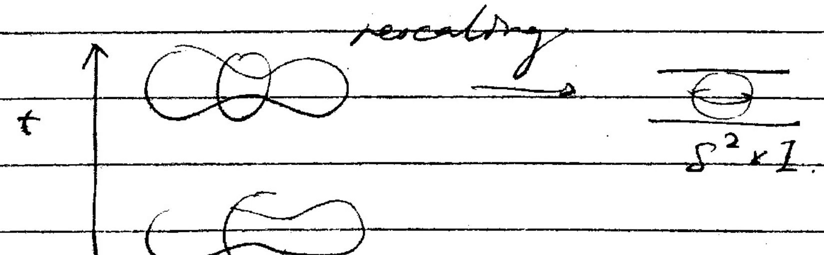
$\Rightarrow |Rm(g_i(0))| \le 1$.

Compactness of RF. if $\{(M_i, g_i(t))\}_{t \in (a,b)}$ $0 \in (a,b)$

s.t. (i) $\sup |Rm(g_i(x,t))| < C$

(ii) $\inf inj(M_i, g_i(0), p_i) > 0$

then $(M_i, g_i(t), p_i) \xrightarrow{CG} (M_\infty, g_\infty(t), p)$.



$n=2$ only blow ups are S^2 or $\mathbb{RP}^2$.

$n=3$. Hamilton - Ivey Pinching

Given $(M^3, g(t))$ $Rm: \Lambda^2 T^*M \longrightarrow \Lambda^2 T^*M$.

has eigenvalues $\lambda_1 \le \lambda_2 \le \lambda_3$.

At any (x,t) s.t. $\lambda_1 < 0$ we have

$$R \ge |\lambda_1| \cdot \left(\log|\lambda_1| + \log(1+t) - 3\right)$$

~~small~~ " small (negative) λ_1 result larger λ_2 or λ_3."

$$\frac{\partial}{\partial t} Rm = \Delta Rm + Rm^2 + Rm^{\#}$$

ODE comparison
$$\begin{cases} \dfrac{d\lambda_1}{dt} = \lambda_1^2 + \lambda_2\lambda_3 \\ \dfrac{d\lambda_2}{dt} = \lambda_2^2 + \lambda_1\lambda_3 \\ \dfrac{d\lambda_3}{dt} = \lambda_3^2 + \lambda_1\lambda_2 \end{cases}$$

Consequences 3D ancient solution.

$(M, g(t))$ $t \in (-\infty, 0]$ have positive solution

(κ-solutions) singularities models of 3D RF.

R large regions closed to κ-solutions.

Def ancient RF with positive curvature $R>0$

κ-noncollapsed at all scales if $B_r(x)$ s.t.

$|Rm| \le r^{-2}$ s.t. $|B_r(x)| \ge \kappa r^n$.

$S^2 \times S^1$ is not a κ-sol.

Classification of 3D models

$Rm > 0$ compact S^3/Γ.

noncompact diffeo to $\mathbb{R}^3$. Bryant

$Rm \geq 0$ $\mathbb{R} \times S^2$.

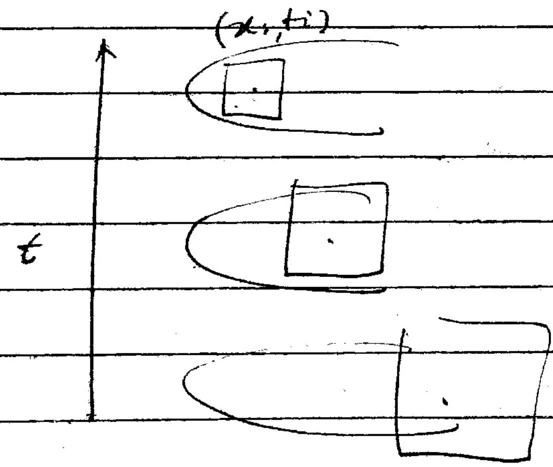
Gradient shrinking solitons.

$(M, g, \nabla f)$ s.t. $Ric + \nabla^2 f - \frac{g}{2} = 0$

$$\Leftrightarrow \quad g(t) = -t \, \phi_t^* g \quad t < 0$$

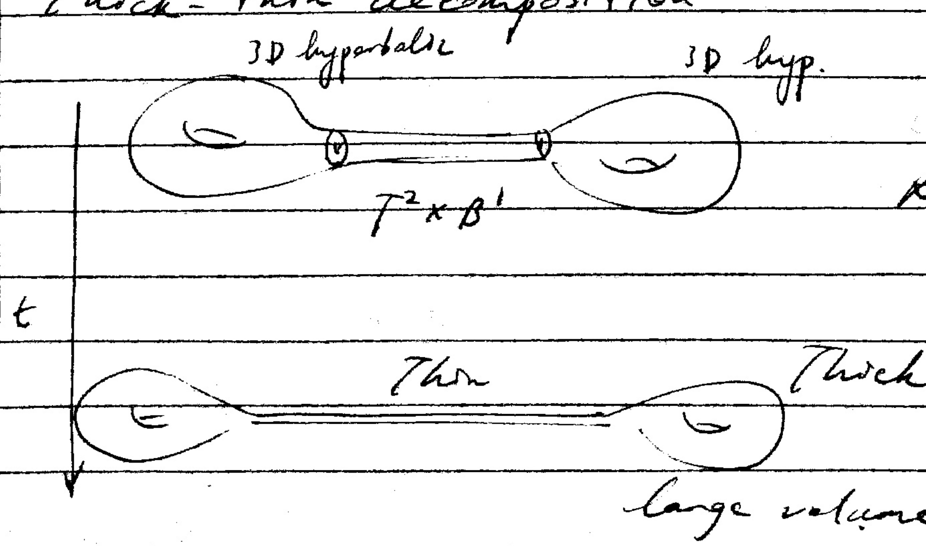
$$\frac{d}{dt}\phi = -\frac{1}{t}\nabla f \quad \phi(-1) = id.$$

Asymptotic soliton of Perelman for κ solution.



Bryant $\rightsquigarrow$ shrinking cylinder

Thick-thin decomposition

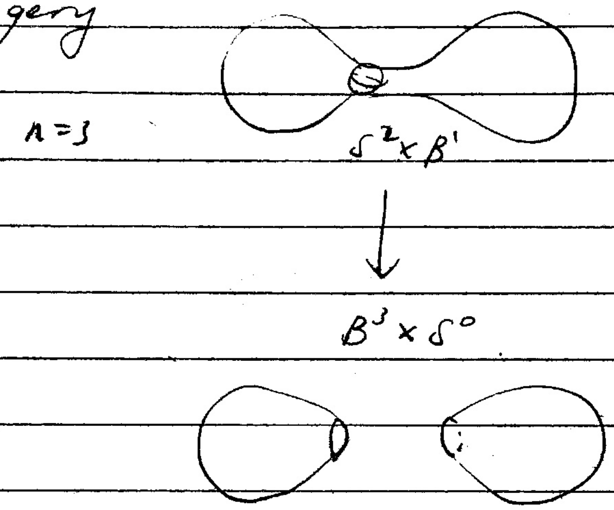


Rescaled flow $\frac{g(t)}{t}$

Why 4D.

h-cobordism in $n \geq 5$.

RF surgery



Q. singularities
   surgery
   long time behavior
   topological applications

Singularities.

Bamler '20, '21 : metric flow, F-limits.
weak limits of RF
tangent flows

partial regularity $X = R \cup S$. $\dim S \leq n-2$

space-time dim

R satisfies $Ric + \nabla^2 f - \frac{g}{2} = 0$ $R > 0$

4D thick-thin decomposition : 4D Einstein orbifolds

some blow up of 4D RF is

- compact shrinkers.
- $\mathbb{R} \times S^3$, $\mathbb{R}^2 \times S^2$
- cone $\mathbb{R}_+ \times N^3$ with $R > 0$

possible applications

11/8 conj. : smooth closed M^4 with indefinite even intersection form satisfies

$$b_2(M^4) \geq \frac{11}{8} |\sigma(M^4)|.$$

$\Rightarrow$ positive answer.

classify simply conn. 4 manifolds up to homeom.

Relation to RF

- compact gradient shrinker, spin $\sigma(M) = 0$
  noncompact. ?
- monotonicity under surgeries $S^3 \times B^1$, $S^2 \times B^2$.
- ALE.
  thick - thin decomp

"Ricci soliton in dim 4 and higher"
Chow, Kotschwar, Munteanu. 2025.

Topics.

① Bamler's theory
② Kähler Ricci flow $n=4$
  - classification of singularity
  - FIK stable. by Naff, Ozuch
  - noncollapsed sing are Type I.
③ examples of solitons.
  Buttsworth $SO(2) \times SO(3)$
  Appleton $U(2)$ invariant.

④ flowing past sing.
- Carson $\mathbb{R}^2 \times S^2$.
- Giannioti - Schulze (conical)

⑤ stability and uniqueness of solitons
- Lai, uniq. of asymptotically cylindrical steadies.

⑥ long time behaviour
Daniels - Orzuch $n=4$ ancient and immortal flows. singular tangent flows.