Sep 16

11/8 conjecture

$\pi_1(X) = \langle 1 \rangle$. topological 4-manifolds.

Thm 1) $N_1 \underset{\text{homeo}}{\cong} N_2 \iff Q_{N_1} \cong Q_{N_2}$ and

$ks(N_1) = ks(N_2) \in H_4(X, \mathbb{Z}_2)$

2) Q not even, Q unimodular and symplectic

$\det Q = \pm 1$.

Q intersection form $H_2 \oplus H_2 \longrightarrow \mathbb{Z}$

$(\alpha, \beta) \longmapsto \langle \alpha \wedge \beta, [M] \rangle$

$\forall (Q, k) \in$ Quadric forms $\times H_4(X, \mathbb{Z}_2)$

$\exists X : Q_N = Q \quad ks(N) = k$.

3) Q even $\Rightarrow \exists x.\ Q_X = Q \quad ks(X) = k$

$\iff k = \frac{\sigma_Q}{8} \pmod 2$.

Thm (Donaldson). Assume Q is definite.

If $Q = Q_M$. M smooth, simply closed, 4-dim.

then $Q = n(1)$ or $n[-1]$

positive def. negative def.

Yang-Mills $\quad YM(D) = \int_M |F_D|^2 \quad SU(2)$.

4D. $F = F^+ + F^-$

$* : \Omega^2 \longrightarrow \Omega^2 \quad *^2 = id \quad \Omega = \Lambda^+ \oplus \Lambda^-$.

$YM(D) = \int |F_D^+|^2 + \int |F_D^-|^2 \geq 8\pi^2 k$

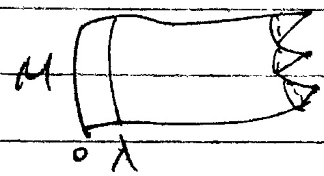
$8\pi^2 k = -c_2(\eta)[M] = -\int_M tr(F \wedge F)$.

~~Def: monopole charge.~~

$\mathcal{M}_{k=1} = \{ *F_D = F_D \} / \text{gauge}$ is 5-dim.

$M \times [0,\lambda] \subseteq \mathcal{M}$.

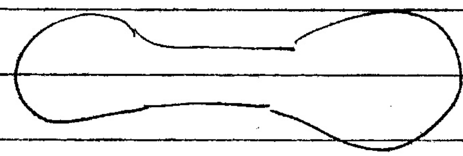
singular $E \cong L \oplus L^{-1}$. L a $U(1)$-bundle.



Cone over $\mathbb{C}P^2$ or $\overline{\mathbb{C}P^2}$.

opposite orientation

For $\mathbb{C}P^2$ $Q = [1]$.



$$Q_{M\#N} = Q_M \oplus Q_N.$$

$M \quad \# \quad N$

e.g. $\#_n\mathbb{C}P^2 \# m\overline{\mathbb{C}P^2}$ $\quad Q = n[1] \oplus m[-1]$.

$ks \in H^4(M, \mathbb{Z}_2) \cong \mathbb{Z}_2$ $\quad$ ks nonzero $\Rightarrow$ no smooth structure

<u>Def.</u> $\text{rank}(Q) = \text{rank}(H)$ $\quad Q: H \times H \longrightarrow \mathbb{Z}$.

$\sigma_M = \text{sign}_Q = \dim H_+^2 - \dim H_-^2$.

- Q_M <u>definite</u> if $Q_M(\alpha,\alpha) > 0$ $\forall \alpha \in H$.
- Q_M <u>indefinite</u> if $\exists\, \alpha_\pm$ s.t. $Q(\alpha_+, \alpha_+) > 0$ $\quad Q(\alpha_-, \alpha_-) < 0$.
- Q_M <u>even</u> / <u>odd</u> if $Q(\alpha,\alpha) = \begin{cases} 0 & \bmod 2 \\ 1 & \bmod 2 \end{cases}$ $\forall \alpha$

Examples.

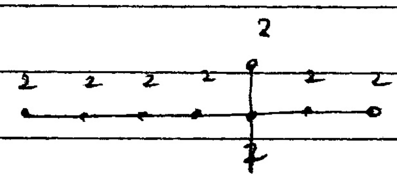
- $Q_{\mathbb{C}P^2} = (1)$
- $Q_{S^2\times S^2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$\begin{tikzcd} & S^2\times S^2 \arrow[dl] \arrow[dr] & \\ S^2 & & S^2 \end{tikzcd}$$

- $Q_{S^2\tilde{\times} S^2} = \begin{bmatrix} 1 & 1 \\ 1 & \end{bmatrix} \cong \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = [+1] \oplus [-1].$

$$\begin{tikzcd} S^2\tilde{\times} S^2 \arrow[d, phantom, "\cong"] \\ \mathbb{C}P^2 \# \overline{\mathbb{C}P}^2 \arrow[d] \\ S^2 \end{tikzcd}$$

- Q_{E_8}



$$\begin{matrix} 2 & 1 & & & & \\ 1 & 2 & 1 & & & \\ & 1 & \ddots & & & \\ & & & 2 & 1 & \\ & & & 1 & 2 & \\ & & & & & 2 \end{matrix}$$

$\sigma = 8$. $\det Q = 1$.

X_{E_8} non smoothable

$X_{E_8} \# X_{E_8} = X_{E_8 \oplus E_8}$ cannot have a smooth str.