Oct 7. Furuta's $^{10}/_{8}$ Theorem

$$b_2(M) = \frac{10}{8}|\sigma(M)| + 2$$

M is a closed connected spin 4-manifold. smooth.
Assume $b_1(M) = 0$ simply connected.
$\longrightarrow$ ~~simply~~ symmetry group of SW eqn

Step 1

L complex line bundle $w_1(L) = c_1(L) \bmod 2$.
twisted. $\Rightarrow$ spin^c structure get a map

$$\Gamma(S^+) \times \mathrm{Conn}(L) \longrightarrow \Gamma(S^-) \times \Gamma(\Lambda^+)$$
$$(\phi, \nabla^A) \longmapsto (D^A\phi, F_A^+ + \phi i \bar{\phi})$$

locally $\nabla^A = d + iA$, $A \in \Omega^1$

M is spin $\Rightarrow$ L is trivial, ∇^A flat connection
We can fix a flat connection to get a map

$$\Gamma(S^+) \times \Omega^1 \longrightarrow \Gamma(S^-) \times \Gamma(\Lambda^+)$$
$$(\phi, A) \longmapsto (D\phi + Ai\phi, F_A^+ + \phi i \bar{\phi})$$

Step 2 Compactness properties $\Rightarrow$ projecting to span of singular values of D of $\leq \lambda$ is finite dimensional

Step 3. As λ increases, $(D+Q)_\lambda$ has a fixed homotopy class. $\Rightarrow$ get a fixed class.

Step 4 The degree of the class + $K(\mathrm{Pin}(2))$
$\Rightarrow b_2 \geq 2k+1$.

$Spin(4) \cong SU(2) \times SU(2)$

$\cong \mathbb{H}^* \times \mathbb{H}^\times$. unit quaternions

irreducible representative from $\mathbb{H}$.

$(q_-, q_+, q_0)v = q_- v q_+^{-1}$ "$_-\mathbb{H}_+$"

$(q_-, q_+, q_0)v = q_+ v q_0^{-1}$ "$_+\mathbb{H}$"

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→ Get 4 representation of $Spin(4)$ $Pin(2)$

$Pin(2) \subseteq \mathbb{H}^\times$

$S^\pm = P_{spin} \times_{\pm} \mathbb{H}$ $\quad T = P_{spin} \times_- \mathbb{H}_+ \cong TM$.

$\Lambda = P_{spin} \times_+ \mathbb{H}_+$

Spin Connection $\nabla^1 : \Gamma(S^+) \longrightarrow \Omega^1 \otimes \Gamma(S^-)$

Clifford multiplication $c: T \otimes S^+ \longrightarrow S^-$.

Dirac $D^1 = c\,\nabla^1 : \Gamma(S^+) \longrightarrow \Gamma(S^-)$

$D^2 = \bar{c}\,\nabla^2 : \Gamma(\tilde{\Lambda}) \longrightarrow \Gamma(\tilde{\Lambda})$

Levi-Civita.

$\bar{c} : \Gamma(T) \otimes \Gamma(\tilde{T}) \longrightarrow \Gamma(\tilde{\Lambda})$

$\alpha \quad \beta \longmapsto \langle \alpha, \beta \rangle \oplus p^+(\alpha \wedge \beta)$

$Pin(2)/S^1 = \{\pm 1\}$.

$D^2(\alpha) = d^* \alpha + d^+ \alpha$.

Coulomb gauge.

$$D^1 + D^2 : \Gamma(S^+ \oplus \tilde{T}) \longrightarrow \Gamma(S^- \oplus \tilde{\Lambda}).$$

$$Q : \Gamma(S^+ \oplus \tilde{T}) \longrightarrow \Gamma(S^- \oplus \tilde{\Lambda})$$

$$(\phi, A) \longmapsto (A i \phi,\ \phi i \bar{\phi})$$

$D + Q = D_1 + D_2 + Q$ = Seiberg-Witten eqn's.

$A \in \Omega^1$ gauge group $\mathrm{Hom}(M, S^1) \cong S^1$.

$j \in Pin(2)$ also fixes $D + Q = 0$

Symmetries of $D + Q$ $\langle S^1 \cong U(1), j \rangle = Pin(2)$.

Extend $D + Q : V \longrightarrow W$.

Sobolev $V = L^2_4(\Gamma(S^+ \oplus \tilde{T}))$ $W = L^2_3(\Gamma(S^- \oplus \tilde{\Lambda}))$

$$\|v\|_V^2 = \|(D^* D)^2 v\|_{L^2}^2 + \|v\|_{L^2}^2$$

$$\|w\|_W^2 = \|(D^* D)^{3/2} w\|_{L^2}^2 + \|w\|_{L^2}^2.$$

<u>Thm</u> $\{v \in V \mid (D + Q) v = 0\}$ is compact.

- Moduli space is compact.
- Weitzenböck formula elliptic estimate on D_A

$$|D^* D \phi|^2 = |\nabla_A^* \nabla_A \phi| + \frac{s}{2} |\phi|^2$$

$\Rightarrow$ if $R \gg 0$ then $\exists\, \varepsilon > 0$

$$\|v\|_V = R \Rightarrow \|(D + Q) v\|_W \geq \varepsilon.$$

$P_\lambda : W \longrightarrow W_\lambda$ $\quad L^2$-projection

$V_\lambda \leq V$ $\quad$ eigenvalues $\lambda' < \lambda$ of D^*D.

$W_\lambda \leq W$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad DD^*$

Then $D + P_\lambda Q : V_\lambda \longrightarrow W_\lambda$ is well-defined a bounded map b/w finite dim vector spaces.

$\Rightarrow \|P_\lambda Q v\|_W < \varepsilon$ $\qquad 1 - P_\lambda \longrightarrow 0$ as $\lambda \longrightarrow \infty$.

<u>Corollary</u> $D + P_\lambda Q$ does not vanish on $S(R) \cap V_\lambda$.

$$
\begin{tikzcd}
S(V_\lambda) \arrow[r, "R"] \arrow[drr, dashed] & V_\lambda \arrow[r, "D + P_\lambda Q"] & \bar{W}_\lambda \setminus \{0\} \arrow[d] \\
 & & S(\bar{W}_\lambda)
\end{tikzcd}
$$

$V_\lambda \cong \mathbb{H}^{m'}$ $\quad \dim(\bar{W}_{\lambda_i}) + \mathrm{ind}(D_i) = \dim(V_{\lambda_i})$ $\quad i = 1, 2$

$M + \mathrm{ind}(D_1) + \mathrm{ind}(D_2) = m'$

$\mathrm{ind}(D_1) = 4k$, $\quad \mathrm{ind}(D_2) = -1 - b_+$.

$\tilde{f} : B(W_\lambda, c) \longrightarrow B(\bar{W}_\lambda, c)$

$e(B(V_\lambda, c)) = \deg(\tilde{f}) \cdot e(B(\bar{W}_\lambda, c))$