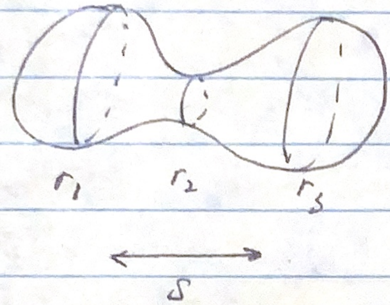
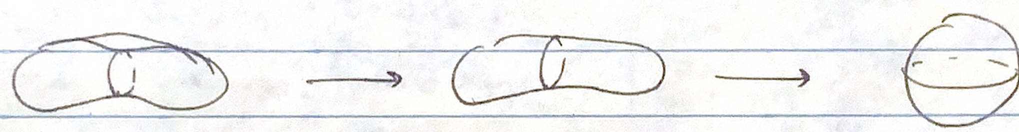
Oct 14 3D Ricci flow.

M



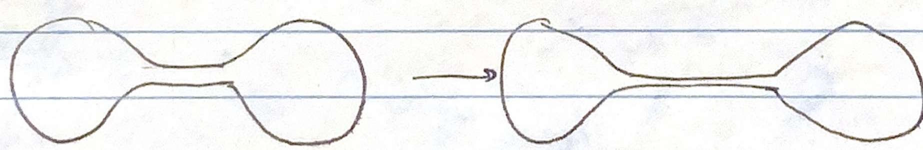
$S^2 \times \mathbb{R}$ with
$g_0 = f(s)^2 g_{S^2} + ds^2$.

Case 1. $r_1 \approx r_2 \approx r_3$.



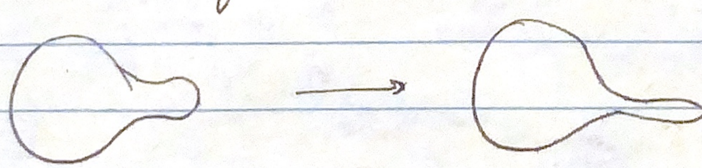
Shrinks to a round point
singularity model $= S^3$.

Case 2 Neck pinch $r_1, r_3 \gg r_2$



$S^2 \times \mathbb{R}$
singularity model

Case 3. Bryant soliton $r_1 \gg r_2, r_3$



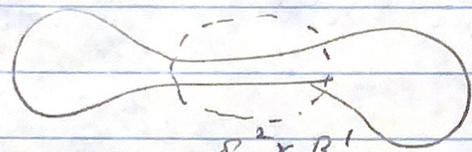
$M_{Bry}.\ g_{Bry} = f(s)^2 g_{S^2} + ds^2$
with $f(s) \sim \sqrt{s}$.

3D surgery

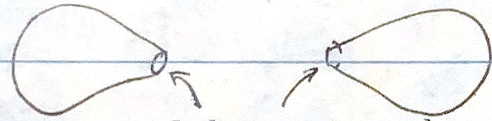
2-surgery on M^3

Remove $S^2 \times B^1$ (cylinder)

Glue in $B^3 \times S^0$ (caps)



$S^2 \times B^1$ cut out



Glue on B^3

Structure theorem for 3D singularity models.

Any singularity model $(M_\infty, g_\infty(t))$ is isom to (modulo rescaling)

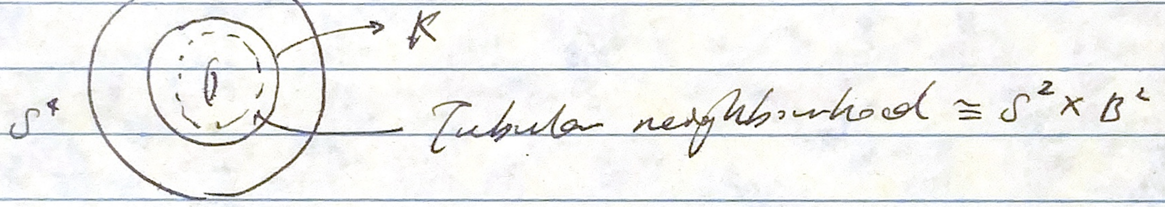
1. quotient of round shrinking $(S^3, (1-4t)g_{S^3})$
2. quotient of round cylinder or $S^2 \times \mathbb{R}/\mathbb{Z}_2$
   $(S^2 \times \mathbb{R}, (1-2t)g_{S^2} + g_{\mathbb{R}})$
3. Bryant soliton.

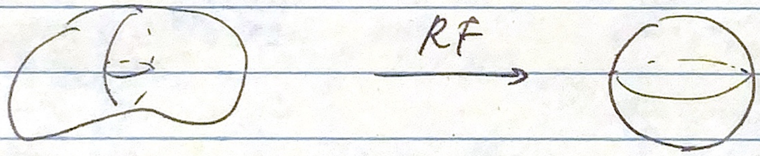
$$M \cong \#(S^3/\Gamma_j) \# m \cdot (S^2 \times S^1) \qquad \Gamma_j \subseteq O(4).$$

Surgery in dim 4.

- 3-surgery on M^4 remove $S^3 \times B^1$ cylinder
glue in $B^4 \times S^0$ caps.
- 2-surgery on M^4 remove $S^2 \times B^2$
glue in $B^3 \times S^1$

Example 2-knot K in S^4.

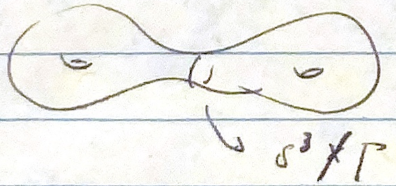


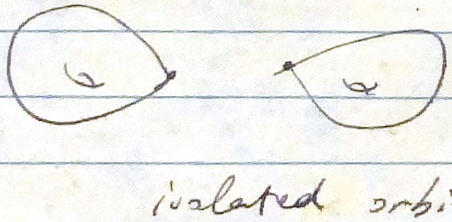


$\pi_1(S^4 \setminus K)$ is more complicated than $\pi_1(S^4)$

Singularity models in 4D.

1. $S^3/\Gamma \times \mathbb{R}$ $\rightsquigarrow$ quotient 3-surgery





isolated orbifold singularities with Γ isotropy

2. $S^2 \times \mathbb{R}^2$ (collapsing knot)

handled with
- regular 2-surgery
- prophylacitc 2-surgery

cut out $S^2 \times \Sigma^2$ glue in $B^3 \times \partial\Sigma$, Σ surface

3. Appleton singularity models

(i) flat cone $\mathbb{R}^4/\mathbb{Z}_2$

(ii) Eguchi-Hanson metric: Ricci flat and asymptotic to $\mathbb{R}^4/\mathbb{Z}_2$.

(iii) Quotient $M_{Bry}/\mathbb{Z}_2$

(iv) $\mathbb{R}P^3 \times \mathbb{R}$

(i) (iv) are gradient shrinking solitons.

(ii) (iii) are not.

4. FZK shrinking solitons

5. BCCD shrinking solitons.

(GSS $\longrightarrow$ PSC $\longrightarrow$ 0 index)

<u>Thm</u> $\begin{pmatrix}\text{Gradient}\\ \text{shrinking}\\ \text{solitons}\end{pmatrix}$ satisfies 11/8 conjecture

pf.

① GSS have positive scalar curvature

② Bochner formula $\quad D^2 = \nabla^* \nabla + \frac{1}{4} K$

$\quad D$ is invertible $\Longrightarrow$ ind $D = 0$

③ $\operatorname{ind} D = -\frac{1}{24} \int_{M^4} p_1(M^4) = -\frac{1}{8}\sigma(M)$

④ $\dfrac{|\sigma(M)|}{b_2(M)} = 0 \leq \dfrac{8}{11}$.

Ricci flow approach to 11/8 conj

$M_1 \to M_2$ under RF with surgery

① M_2 is simple enough to show that

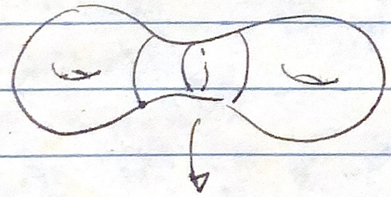
$$\frac{|\sigma_1(M_2)|}{b_2(M_2)} \leq \frac{8}{11}$$

② Show that the ratio is non decreasing under the surgeries.

$$\frac{|\sigma_1(M_1)|}{b_2(M_1)} \leq \frac{|\sigma_1(M_2)|}{b_2(M_2)} \leq \frac{8}{11}$$

Known: 2-surgery and 3-surgery preserve the inequality

Abstract 3-surgery.



M_1 $\qquad\qquad$ M_2

S^3/Γ is a rational homology S^3.

$$H_\bullet(S^3/\Gamma; \mathbb{Q}) \cong H_\bullet(S^3; \mathbb{Q})$$

Mayer-Victoris sequence

$$0 = H_2(S^3/\Gamma) \longrightarrow H_2(M_1) \oplus H_2(M_2) \xrightarrow{i_1 + i_2} H_2(M)$$

$$\to \underset{\overset{\|}{0}}{H_1(S^3/\Gamma)} \longrightarrow \cdots$$

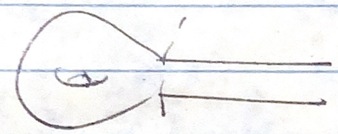
$$\Rightarrow \quad b_2(M) = b_2(M_1) + b_2(M_2)$$

$$\sigma(M) = \sigma(M_1) + \sigma(M_2)$$

Prophylactic 2-surgery

WTS: b_2 is nonincreasing, σ is preserved

so $|\sigma|/b_2$ is nondecreasing.



W $\quad$ $\Sigma^2\times S^2$ $\quad$ Σ surface with bdry.

$\partial W = \partial\Sigma\times S^2.$

Mayer-Victoris sequence

$$H_2(\partial\Sigma\times S^2) \cong H_2(\partial W) \xrightarrow{(i_*,\, j_*)} H_2(\Sigma^2\times S^2)\oplus H_2(W)$$
$$\longrightarrow H_2(M^4) \longrightarrow 0$$

$i_*: H_2(\partial\Sigma\times S^2) \longrightarrow H_2(\Sigma^2\times S^2)$ is an isom.

$\Rightarrow b_2(M^4) \geq b_2(W) \geq b_2(\tilde{M}^4)$

Next.

$$H_2(\partial\Sigma\times B^3)\oplus H_2(W) \xrightarrow{\alpha} H_2(\tilde{M}^4) \xrightarrow{\beta} H_1(\partial\Sigma\times S^2).$$
$$\xrightarrow{(i_*,\, j_*)} \underbrace{H_1(\partial\Sigma\times B^3)}_{0}\oplus H_1(W).$$

i_* is an isomorphism.

$\mathrm{im}\,\alpha = \ker\beta. = H_2(\tilde{M}^4)$