Set up

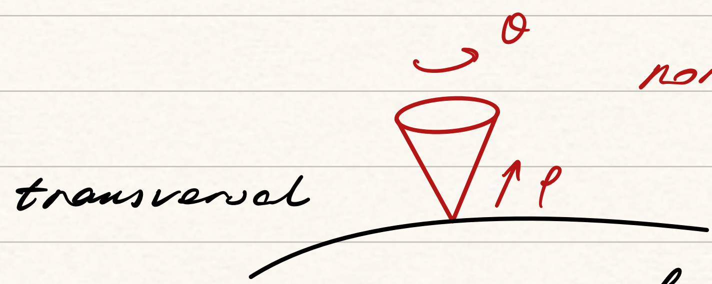
(M^4, g)

$\Sigma^2 \subseteq M^4 \qquad M - \Sigma$ smooth

Coordinates

local $(y_1, y_2, x_1, x_2) \qquad y_1 = y_2 = 0$ on Σ

polar (ρ, θ, x_1, x_2)

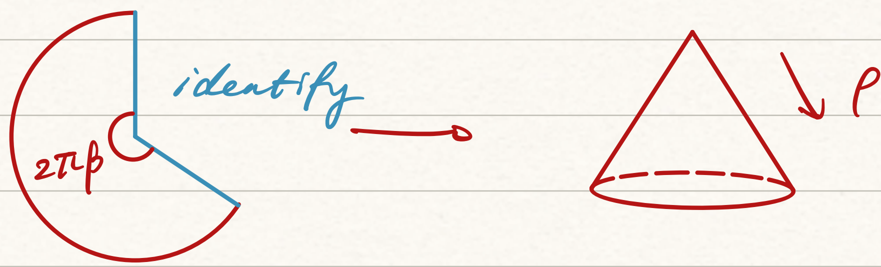


normal cone with angle $2\pi\beta$

$(\rho, \theta) \qquad d\rho^2 + \beta^2 d\theta^2$.

horizontal base $\Sigma^2 \quad (x^i)$

$w_{jk}\, dx^j\, dx^k$



Edge-Cone Metric

Normal bundle near Σ: $\quad N\Sigma \cong \Sigma^2 \times \mathbb{R}^2$

$$d\rho^2 + \beta^2\rho^2 d\theta^2 + g_\Sigma(x)$$

S^1-connection $\qquad S^1 \hookrightarrow N\Sigma \xrightarrow{\pi} \Sigma$

Model cone metric

$$\bar{g} = d\rho^2 + \beta^2\rho^2 (d\theta + v)^2 + g_\Sigma$$

$$= d\rho^2 + \beta^2\rho^2 (d\theta + \underbrace{u_j(x)\, dx^j}_{\text{smooth conn. 1-form}})^2 + w_{jk}\, dx^j\, dx^k$$

Edge-cone metric

$$g = \bar{g} + \underbrace{\rho^{1+\varepsilon} h}_{} \quad \text{on } M \setminus \Sigma$$

pertubation, h symmetric 2-tensor

Einstein edge-cone metric: g Einstein on $M \setminus \Sigma$

e.g. $\beta < \frac{1}{3}$ Kähler-Einstein

$\beta = \frac{1}{p}$, $p \geq 2$, quotient of non-singular Einstein $X/\mathbb{Z}_p$ w/ codim 2 fixed points

4D Hitchin-Thorpe ineq.

- smooth compact Einstein

$$2\chi(M) \pm 3\tau(M) \geq 0$$

pf. Gauss-Bonnet type integrals

$$\chi(M) = \frac{1}{8\pi^2} \int_M \left(\frac{s^2}{24} + |W|^2 - \frac{|\mathring{Ric}|^2}{2} \right) d\mu$$

$$\tau(M) = \frac{1}{12\pi^2} \int_M \left(|W_+|^2 - |W_-|^2 \right) d\mu$$

0 Einstein

$$\Rightarrow \quad LHS = \frac{1}{4\pi^2} \int_M \left(\frac{s^2}{24} + 2|W_\pm|^2 - \frac{\cancel{|\mathring{Ric}|^2}}{2} \right) d\mu$$

Correction needed for χ and τ w/ edge-cone singularity

• edge-cone singularity

$$2\chi(M) \pm 3\tau(M) \geq (1-\beta)\left(2\chi(\Sigma) \pm (1+\beta)[\Sigma]^2\right)$$

where M^4 smooth compact

Σ^2 smooth embedded compact oriented

(M, Σ) admit Einstein edge cone metric g w/ angle $2\pi\beta$

Thm 2.1 & 2.2

$$\chi(M) - (1-\beta)\chi(\Sigma) = \frac{1}{8\pi^2}\int_M \left(\frac{s^2}{24} + |W|^2 - \frac{|\mathring{Ric}|^2}{2}\right) d\mu$$

$$\tau(M) - \frac{1}{3}(1-\beta^2)[\Sigma]^2 = \frac{1}{12\pi^2}\int_M \left(|W_+|^2 - |W_-|^2\right) d\mu$$

self-intersection of cohomology class in $H^2(M) \overset{PD}{\cong} H_2(M)$

$= \langle [\Sigma] \cup [\Sigma], [M] \rangle$

$$\underbrace{2(1-\beta)\chi(\Sigma) \pm (1-\beta^2)[\Sigma]^2}_{\shortparallel}$$

$$\Rightarrow 2\chi(M) \pm 3\tau(M) - \boxed{def_\pm(\Sigma, \beta)}$$

$$= \frac{1}{4\pi^2}\int_M \left(\frac{s^2}{24} + 2|W_\pm|^2 - \frac{|\mathring{Ric}|^2}{2}\right) d\mu$$

Step 1. Metric - independent

Lemma 2.1 it suffices to check (★) for one edge-cone metric

pf. Def of edge cone metric controls decay rate of the integrand.

Let $\dot{g} = g' - g$ w/ same angle

$g_t = g + t\dot{g} + O(t^2)$

edge-cone $g = \bar{g} + \rho^{1+\varepsilon} h$ almost smooth so

∂g smooth $+ O(\rho^{\varepsilon})$, $\partial^2 g \sim O(\rho^{-1+\varepsilon})$

$\Rightarrow |R| \sim O(\rho^{-1+\varepsilon})$

$\Rightarrow |\phi_t|, |\psi_t| \leq |R| \cdot |\nabla \dot{g}| \quad C\rho^{-1+\varepsilon}$

$$\left| \frac{d}{dt} \int_{M_\delta} \Phi_t \right| = \int_{\partial M_\delta} \phi_t \leq C_1 \delta^{-1+\varepsilon}$$

$$= C_2 \delta^{\varepsilon} \xrightarrow{\delta \to 0} 0$$

Gauss - Bonnet integrand

Step 2 Simplification to a Ricci form

- Let $\Upsilon = \left(\frac{s^2}{24} + 2|W_+|^2 - \frac{|\mathring{Ric}|^2}{2} \right) d\mu$

g_0 smooth metric $\quad g$ edge-cone metric

Then

$$\int_M \Upsilon_g - \boxed{\int_M \Upsilon_{g_0}} = \int_M \boxed{\Upsilon_g - \Upsilon_{g_0}} = -4\pi^2 \, def_+(\Sigma, \beta)$$

$= 4\pi^2 (2\chi + 3\tau)(M)$

support in a tubular neighbourh V of Σ

- Kähler $\Rightarrow |W_+|^2 = \frac{s^2}{24}$

$$\Upsilon = \frac{1}{2} \left(\frac{s^2}{4} - |\mathring{Ric}| \right)^2 d\mu = \rho \wedge \rho$$

Ricci form

supp of $\rho \wedge \rho$

zero measure sing. set.

$$LHS = \int_{V - \Sigma} \rho_g \wedge \rho_g - \rho_{g_0} \wedge \rho_{g_0}$$

$$\overset{NTS}{=} -4\pi^2 (1-\beta) \left(2\chi(\Sigma) + (1+\beta)[\Sigma]^2 \right)$$

Step 3. Prove (✯) for a special case

Prop 2.1 proves (✯) by constructing a specific model metric and compute the defect term explicitly.

- construct a rotationally symmetric Kähler metric

$$\Phi(\zeta, \bar{\zeta}) = F(|\zeta|^2) = F(t)$$

Kähler potential   auxilliary func.

$$\omega = i\partial\bar{\partial} F(|\zeta|^2) = i f(|\zeta|^2)\, d\zeta \wedge d\bar{\zeta}$$

local Kähler form

Def. $\displaystyle \frac{dF}{dt} = \frac{1}{t}\int_0^t f(x)\,dx$, $\quad f(x) = \begin{cases} \frac{1}{\beta} & t \le \frac{1}{2} \\ t^{\beta-1} & t \ge 1 \end{cases}$

f controls cone angle

ϖ normal bundle $E \to \Sigma$

$$\left.\begin{aligned} \omega &= \lambda \varpi^* \alpha + i\partial\bar{\partial}(\beta^{-2} t^{\beta}) \rightsquigarrow g \\ \omega_0 &= \underbrace{\lambda \varpi^* \alpha}_{\text{form on } E} + i\partial\bar{\partial} F(t) \rightsquigarrow g_0 \end{aligned}\right\} \text{agree for } t \ge 1$$

area form of Σ

locally

$$\begin{aligned} \omega &= i t^{\beta-1} d\zeta \wedge d\bar{\zeta} + i(\lambda - \frac{\kappa}{\beta} t^{\beta}) dz \wedge d\bar{z} + O(|z|) \\ \omega_0 &= i f(t) d\zeta \wedge d\bar{\zeta} + i(\lambda - \kappa t F'(t)) dz \wedge d\bar{z} + O(|z|) \end{aligned}$$

normal fiber $E \ni (x, \zeta)$

$$\int_{V-\Sigma} \rho_g^2 - \rho_{g_0}^2 \overset{\text{Stoke}}{=} \boxed{-\int_{S_\varepsilon} \phi \wedge (2\rho_0 + d\phi)}$$

where $\rho - \rho_0 = d\phi$ / $\phi = -\bar\partial\partial \log\left(\frac{V}{V_0}\right)$

can compute explicitly

ε width of tubular neighbourhood

$$= 4\pi^2(1-\beta)\left(2\chi(\Sigma) + (1+\beta)[\Sigma]^2\right) + \mathcal{O}\left(\varepsilon^{2\min(1,\beta)}\right)$$

then let $\varepsilon \to 0$

Rmk. If Σ non-oriented, but M oriented, then we need to pass the computation of $\chi(\Sigma)$ to the oriented double cover.

Prop 2.2 (M, J) almost complex 4-manifold
orientation induced by J
Σ totally real w.r.t J

$$\tau(M) + \frac{1}{3}(1-\beta^2)\chi(\Sigma) = \frac{1}{12\pi^2}\int_M \left(|W_+|^2 - |W_-|^2\right) d\mu$$

§ 3. G-index proof $\quad M = X/G$

Set up

finite G acting on compact oriented X.
preserving the metric & orientation

Let $x \in X^g$, fixed point set of g, the linearized action decomposes as rotations.

$$g|_{T_x X} = \begin{pmatrix} R(\alpha) & 0 \\ 0 & R(\beta) \end{pmatrix}$$

$H^2(X, \mathbb{R}) = H^+ \oplus H^-$
$\forall g \in G$, g_* induced action on $H^2(M; \mathbb{R})$
Set $\tau(g, X) = tr(g_*|_{H^+}) - tr(g_*|_{H^-})$

Orbifold signature

$g = 1 \quad \tau(1, X) = \text{signature}$
$g \neq 1 \quad X^g = \text{fixed point set of } g$

$$= \bigsqcup x_j \cup \bigsqcup \hat{\Sigma}_k$$

isolated $\alpha, \beta \neq 0$ $\qquad$ compact surfaces one of α or $\beta = 0$

$$\Rightarrow \tau(g, X) = -\sum_j \cot\frac{\alpha_j}{2} \cot\frac{\beta_j}{2} + \sum_k \left(\csc^2\frac{\theta_k}{2}\right)[\Sigma_k]^2$$

Averaging over G yields the orbifold signature

$$\tau(M) = \frac{1}{|G|}\left(\tau(X) + \sum_{g \neq 1} \tau(g, X)\right)$$

$\beta = \frac{1}{p} \quad p \in \mathbb{Z}_{\geq 0}$

Apply this to $G = \mathbb{Z}_p$ and use $\sum \csc^2(k\pi/p) = \frac{p^2-1}{3}$

$$\Rightarrow \quad \tau(M) - \frac{1}{3}\left(1 - \frac{1}{p^2}\right)[\Sigma]^2 = \frac{1}{p}\tau(X)$$

$$= \frac{1}{12\pi^2}\int_M |W_+|^2 - |W_-|^2 \, d\mu$$

§4. Einstein

Since $2\chi(M) \pm 3\tau(M) - \mathrm{def}_\pm(\Sigma, \beta)$

$$= \frac{1}{4\pi^2}\int_M \left(\frac{s^2}{24} + 2|W_\pm|^2 - \frac{\cancelto{0}{|\mathring{Ric}|^2}}{2}\right) d\mu$$

we know Einstein $\Rightarrow$ RHS ≥ 0 so

$$2\chi(M) \pm 3\tau(M) \geq \mathrm{def}_\pm(\Sigma, \beta) \qquad (\star)$$

w/ equality $\Leftrightarrow$ $s = 0$ and $W_\pm \equiv 0$

$\Leftrightarrow$ $\Lambda^\pm$ flat

(Einstein) $\Leftrightarrow$ locally hyper Kähler

(4D) $\Leftrightarrow$ Ricci flat + locally Kähler

- small angle : $\beta_i \to 0$ Coro 4.2

$$2\chi(M) \pm 3\tau(M) \geq 2\chi(\Sigma) \pm [\Sigma]^2$$

- large angle : obstruction to Einstein metric

$\frac{1}{\beta^2}(\star^\pm)$ let $\beta \to \infty$ $\Rightarrow \begin{cases} 0 \geq -[\Sigma]^2 \\ 0 \geq [\Sigma]^2 \end{cases} \Rightarrow \boxed{[\Sigma]^2 = 0}$

$\frac{1}{4\beta}\left((\star^+) + (\star^-)\right) \Rightarrow \boxed{0 \geq -\chi(\Sigma)}$

§5.

- families of gravitational instanton

self-dual : $W_- = 0$

complete, non-compact, Ricci-flat M^4.

- hyperbolic ansatz

$U \subseteq \mathbb{H}^3$ 3D hyperbolic space

$V : U \to \mathbb{R}$ harmonic w.r.t. hyperbolic metric defines a connection Θ 1-form of a principal $U(1)$-bundle via $d\Theta = * dV$.

Set $g_0 = Vh + V^{-1}\Theta^2$ on $\begin{array}{c} P \\ \downarrow \\ U \end{array}$

- Consider potential $V = \beta^{-1} + \underbrace{\sum_{j=1}^{n} G_{p_j}}$

$\beta > 0$ const.

superposition of Green's functions in $\mathbb{H}^3$

p_j distinct pts.

By conformal rescaling $g = \beta (\operatorname{sech}^2 s) g_0$

dist from arbitrary base pt in $\mathbb{H}^3$.

- compactification

$$M = \mathcal{P} \cup \Sigma \cup \{\hat{p}_j\} \cong n\,\mathbb{C}P^2 = \mathbb{C}P^2 \# \cdots \# \mathbb{C}P^2$$

interior

added 2-sphere comes from sphere at infinity "cone singularity"

added to compactify fibers over p_j "smooth"

Near p_j: $V \sim e^{-2\varrho}$,

$g_0 \sim$ Gibbons–Hawking near isolated center

$\hat{p}_j$ corresponds to non singular nut

Hitchin–Thorpe says

$$\frac{1}{12\pi^2}\int_M |W_+|^2\, d\mu = \tau(M) - \frac{1}{3}(1-\beta^4)[\Sigma]^2 = \frac{n(2+\beta^2)}{3}$$

Example: $n=1$ $\quad V = \beta^{-1} + (e^{2\varrho}-1)^{-1}$

$$\tilde{g} = \frac{1}{4\beta}\Big((2-\beta)\cosh\varrho + \beta\sinh\varrho\Big)^{-2} g_0$$

is Einstein w/ const $= \frac{3}{2}\beta^2(2-\beta)$ $\qquad 0<\beta<2$

$\beta = 1$ Fubini–Study on $\mathbb{C}P^2$

$\beta \to 0$ Taub–NUT on $\mathbb{R}^4$

$\beta \to 2$ Eguchi–Hanson on TS^2.