Dec 2 Entropy and heat kernel on a RF background
Compactness theory of the space of super Ricci flow $\partial_t g \geq -2\,Ric_g$

Structure theory of noncollapsing limits of RFs.
R. Bamler.

Recall $(M^n, g_t)_{t\in[0,T)}$ compact RF.
$|Rm| \to \infty$ as $t \to T$.
Take $\{(x,t)\} \subseteq M \times [0,T)$, $t_i \to T$
$|Rm|(x_i, t_i) \sim \sup_{[0,t_i]} \sup_M |Rm|$

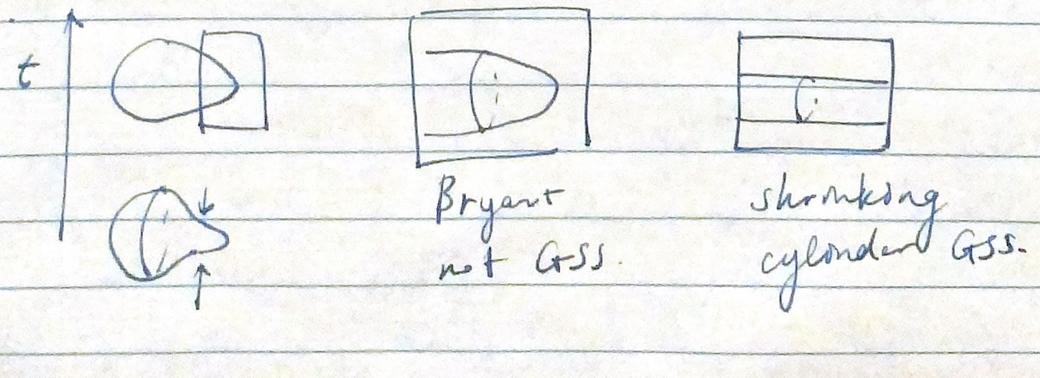
rescale $(g_i)_t = \lambda_i\, g_{\lambda_i^{-1}(t-t_i)} \xrightarrow[\text{sing. model}]{} g_\infty$ on $(-\infty, 0]$

If $\sup_t \sup_{M\times\{t\}} |Rm|(T-t) < \infty$, Type I sing (Naber) '10

Enders - Müller, Topping '12, limit always as a gradient shrinking soliton.

Hamilton are singularity of RF in general related to GSS?

Ex.



~~Bal~~ Bamler '20

- view {Ricci flow} $\subseteq$ {metric flows}
- $\mathbb{F}$ distance on metric flows
- compactness theorem w.r.t $\mathbb{F}$-distance for super Ricci flow
- improved convergence and regularity in {RF}.

Metric flow

a metric flow over $I \subseteq \mathbb{R}$ is a tuple

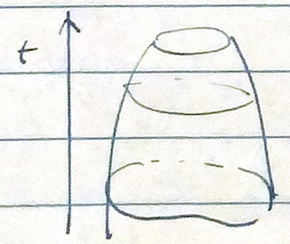
$$(\mathcal{X}, t, (d_t)_{t \in I}, (\nu_{x,s})_{x \in \mathcal{X}, s \in I, s \leq t(x)})$$

space time points

$t: \mathcal{X} \to I$ time function

$(\mathcal{X}_t, d_t)$ complete sep metric space

conjugate heat kernel



$\nu_{x,s}$ satisfies

- $\nu_{x,s} \in \mathcal{P}(\mathcal{X}_s)$ probability measure
- $\nu_{x,t=t(x)} = \delta_x$
- if $t_1 \leq t_2 \leq t_3 = t(x)$ then
$$\nu_{x,t_1} = \int_{\mathcal{X}_{t_2}} \nu_{\cdot,t_1} \, d\nu_{x,t_2}$$
- grad est for heat equation

Ex. $I = \{t\} \subseteq \mathbb{R}$ complet sep. metric space

Ex. $(M, g_t)_{t \in I}$ RF.

metric flow $(M \times I,\ t = \mathrm{proj}_I,\ d_{g_t},\ \nu_{x,s})$

on (M, g_t) we have $K(x,t,y,s)$ for $t > s$

$$(\partial_t - \Delta_x) \int K(x,t,y,s) f(y)\, d_g(y) = 0$$

$$\lim_{t \to s} K(x,t,y,s) = \delta_y.$$

Formally. $\iint \left[(\partial_t - \Delta_x)\, u(x,t)\right] v(x,t)\, d\mu\, dt.$

$$= \iint u(x,t) \underbrace{(-\partial_t - \Delta_x + R)}_{\text{conj. heat op. } \square^*} v(x,t)\, d\mu\, dt$$

$= -R\, d\mathrm{vol}.$

and conj. heat kernel $K(x,t,\cdot,s)$ $s < t$

centered at (x,t).

Note $\int \square u \cdot v = -\int u \cdot \square^* v = \frac{d}{dt} \int u v$

if $\square^* v \Rightarrow \frac{d}{dt} \int v = 0$ i.e. $\nu_{x,t} \in P(\mathcal{X}_s)$

Application to RF.

$(M^*, g_t)_{t\in[0,T]}$, $\tau_i \to 0$, $(M, \tau_i^{-1} g_{T, t\to T})_{t\in[-\tau, T, 0)}$,

$(\nu_{x; T, t\to T})_{t\in[-\tau, T, 0)}$ $(\star)$

Assume $\mathcal{N}_{x_i}(\tau_0) \geq -Y_0$ $\frac{d}{dt}(\tau N) = \mu \leq 0$ $\frac{d}{d\tau} N \leq 0$

$(\star) \xrightarrow{\mathbb{F}, \mathcal{C}} (\mathcal{X}, (\mu_t^\infty)_{t<0})$ $N(\tau) = \text{const}$ for all τ.

Thm $\mathcal{X}_{<0} = \underbrace{\mathcal{R}}_{} \sqcup \underbrace{\mathcal{S}}_{}$.

regular ; singular part.
smooth RF
spacetime

and $\dim_{M^*} \mathcal{S} \leq \frac{n-2}{2}$ (t count twice).

Tangent flow, parabolic rescale about any $x \in \mathcal{X}_{<0}$
$\longrightarrow$ tangent flow

- on $\mathbb{R}$, all tangent flow are $\mathbb{R}^n$.

Thm (Filteration)

$$S^0 \subset S^1 \subset \cdots \subset S^{n-2} = S.$$

(a) $\dim_{M^*} S^k = k$

(b) $\forall x \in \mathcal{X}_{<0} \setminus S^{k-1}$, we have a tangent flow that is a metric soliton, and $\mathcal{X}'_{<0} \cong \mathcal{X}''_{<0} \times \mathbb{R}^k$.

Thm on the regular part $\mathcal{R}$. $Ric + \nabla^2 f + \frac{1}{2t} g = 0$
on the singular part $\mathcal{S}$, there is a singular space (X^n, d) s.t. $(\mathcal{X}_t, d_t) = (X^n \times \{t\}, |t|^{1/2} d)$
and $\mathcal{R} = \mathcal{R}_X \times (-T_\infty, 0)$.

Rmk. In dim 4, (X, d) is an orbifold with isolated singularities

For large time

$(M, (g_t)_{t \geq 0})$ model RF, if $\{(x_i, t_i)\} \subseteq M \times [0, \infty)$
$\mathcal{N}_{x_i, t_i}(\frac{1}{2} t_i) \geq -Y_i > -\infty$. As $t \to \infty$, metric flow
admits a converging subsequence to $(\mathcal{X}, (\nu_{x,t})_{t \in [-1,1]})$
s.t. $\mathcal{X}_{<0}$ comes from (X^n, d) singular space with
$Ric = -\frac{1}{2} g$ on $\mathcal{R}_X$.

$$M = M_{thick} \cup \underbrace{M_{thin}}_{\mathcal{N} < -Y}$$

$\mathbb{F}$ - distance metric flow

Def. (correspondance $\mathfrak{C}$)

A correspondance between metric flows $\mathcal{X}^1, \mathcal{X}^2$ on $I \subset \mathbb{R}$ is a triple $\mathfrak{C}\big(\underbrace{(Z_t, d_t^Z)_{t \in I}}_{\text{metric spaces}}, (\varphi_t^1)_{t \in I}, (\varphi_t^2)_{t \in I}\big)$

$\varphi_t^i : \mathcal{X}_t^i \to Z_t$ iso embedding

s.t.

<u>Def</u> ($\mathbb{F}$-distance in $\mathfrak{C}$)

$$d_{\mathbb{F}}^{\mathfrak{C}, I}(\mathcal{X}^1, \mathcal{X}^2) = \inf \{ r > 0 : \text{on } I \setminus E, \ |E| < r^2$$

and all couplings in $P(\mathcal{X}^1, \mathcal{X}^2)$ between μ_t^1, μ_t^2 }

$$\int_{\mathcal{X}_t^1 \times \mathcal{X}_t^2} d_{W_1}^{Z_s} \left((\varphi_s^1)_* \nu^1_{x^1; s}, (\varphi_s^2)_* \nu^2_{x^2; s} \right) dq_t(x^1, x^2) \leq r$$

$\forall t, s$

<u>Def</u> $\mathbb{F}$-distance between $\mathcal{X}^1, \mathcal{X}^2$

$$d_{\mathbb{F}}^{I}(\mathcal{X}^1, \mathcal{X}^2) = \inf_{\mathfrak{C}} d_{\mathbb{F}}^{\mathfrak{C}, I}$$

<u>Thm</u> $(\mathbb{F}^I, d_{\mathbb{F}}^I)$ is a metric space

<u>Thm</u> $\{\text{super RF } \partial_t g \geq -2\mathrm{Ric}\} \subseteq \mathbb{F}^I$

is subseq. compact

Given a sequence $\exists \mathfrak{C}$ s.t. $(\mathcal{X}^i) \xrightarrow{\mathbb{F}, \mathfrak{C}} (\mathcal{X}^\infty)$

also satisfies some extra entropy properties.