Feb 12 Q. Which gradient Ricci shrinkers are stable critical points?

$$\nu(g) = \inf \left\{ W(g,f,\tau) \;\middle|\; f \in C_c^\infty(M),\ \tau > 0,\ \frac{1}{(4\pi\tau)^{n/2}} \int_M e^{-f} = 1 \right\}$$

$$W(g,f,\tau) = \frac{1}{(4\pi\tau)^{n/2}} \int_M e^{-f}\Big(\big(\tau|\nabla f|_g^2 + scal_g\big) + f - n\Big)$$

$$\delta\nu = \frac{1}{(4\pi\tau)^{n/2}} \int_M \Big\langle e^{-f}\Big(\tau\underbrace{\Big(-Ric - \nabla^2 f\Big) + \frac{g}{2}}\Big), \delta g \Big\rangle_g$$

$$\nabla^2 f + Ric_g - \frac{g}{2\tau} = 0$$

__Def.__ (g,f) is stable if $\delta^2\nu < 0$.

e.g. $\mathbb{C}^2$, $\mathbb{CP}^1 \times \mathbb{C}^2$ are stable
FIK stable (NO 25)
BCCD $Bl_1(\mathbb{CP}^1 \times \mathbb{C})$ unstable

$$\delta^2\nu = \frac{1}{(4\pi)^{n/2}} \int_M \langle N_f h, h\rangle_g \, e^{-f} \, dvol_g$$

$$h = \delta g, \quad N_f h = \frac{1}{2} L_f h + div_f^*(div_f h) + \frac{1}{2}\nabla^2 v_h - \Xi(h)\, Ric$$

where ① $\mathrm{div}_f = \mathrm{div}(h) - h\nabla_f$,

② v_H is the unique solution of

$$(\Delta - \nabla_{\nabla f}) v_H + \frac{1}{2} v_H = \mathrm{div}_f(\mathrm{div}_f h), \quad \int_M v_H e^{-f} = 0$$

for h sufficiently regular. $v_H \in H^{1,2} \cap C^\infty$.

and ③ $$\Xi(h) = \frac{\int_M \langle Ric, h\rangle_g e^{-f} \, dvol_g}{\int_M scal_g \, e^{-f} \, dvol_g}$$

By gauge fixing, we can get rid of the last several terms in $\delta^2 g$.

Gauge fixing.

① $\mathrm{div}_f(h) = 0$

② $h \perp_{L^2_f} Ric$

① is given by $Sym^2(T^*M) = \underbrace{Im(\mathrm{div}_f^*)}_{\substack{\text{generates a Killing v.f.} \\ \text{will not change the RF shrinker}}} \oplus Ker(\mathrm{div}_f)$

② since $L_f Ric = \frac{1}{2\tau} Ric$.

$N_f Ric = \frac{1}{4\tau} Ric - Ric$.

$N_f h = \frac{1}{2} L_f h, \quad L_f = \nabla_f^* \nabla_f$

$= \frac{1}{2} \Delta_f h + Rm(h, -)$

Instability reduces to find $h : \langle L_f h, h\rangle > 0$.

Thm 1 BCCD has $\delta^2 \nu > 0$

(1) There is $h \in H^2_f$, $h \perp_{L^2_f} Ric$, $div_f h = 0$ s.t. $\langle L_f h, h \rangle > 0$

(2) There is a nonvanishing ancient solution

$$\partial_t h = L_f h$$

where $\|h\|_{L^2_f} < \infty$ uniformly

Thm 3 (M^4, g, J, f) $\quad Ric + \nabla^2 f = \frac{\lambda}{2} g$

if $\lambda \leq 0$ for all $h \in H_f^2$, then $\langle L_f h, h \rangle \leq 0$.

Thm 5 (M^4, g, f) orbifold singularity at a point with symmetries in $SU(2)$.
Let λ be eigenvalue of the weighted self-dual
then orbifold pt is stable if $\lambda < 0$
semistable $\lambda = 0$
unstable $\lambda > 0$

Let $S \in \Lambda_g^+ \otimes \Lambda_g^-$, $tr(S) \in T^*M \otimes T^*M$.

$$h = ug + tr(S)$$

$$L_f h = (\Delta_f u) g + 2u \, Ric + tr\left(\Delta^L_{H,f,0} S + S \circ \left(W^+ + \lambda - \frac{scal}{3} Id_{\Lambda_g^+} \right) \right)$$

where $\Delta^L_{H,f,0} = -\left(d^L \delta^L_{f,0} + \delta^L_{f,0} d^L \right)$ and

$$\delta^L_{f,0} = e^f \delta^L e^{-f}.$$

left $d^L : \Lambda^k \otimes \Lambda^l \longrightarrow \Lambda^{k+1} \otimes \Lambda^l$

$$d^L(\alpha \otimes \beta) = d\alpha \otimes \beta$$

On Kähler M^4, h decomposes as $h_I + h_A$
J-invariant part

$$S = S_I + S_A = \gamma \otimes \omega + S_A$$

self-dual anti-self dual.

Then $L_f h_I = (\Delta_{H,f,0} + \lambda) \gamma \circ \omega$
$= tr(\Delta_{H,f,0} + \lambda) S_I$.

and $L_f h_A = \operatorname{tr}(\Delta^L_{H,f} \circ S_A) + (\lambda - \frac{scal}{2}) h_A$.

<u>Lemma</u> (M, g, f, J) Kähler-Ricci shrinker
If $\dim \mathcal{H}_f^{1,1} \geq 2$, then (M, g, f, J) is unstable

harmonic

BCCD: $(M^4, J) = (Bl_1(\mathbb{CP}^1 \times \mathbb{C}), J)$.

$\mathcal{H}_f^{1,1} = \{ \omega \in \Omega^{1,1} \mid d\omega = \delta_f \omega = 0 \}$.

Ref. 2004. "Gaussian density and stability of some Ricci Solitons" (Cao, Hamilton, Ilmanen)
variational formulae
2011. "On the linear stability of K-R solitons" (Hall, Murphy)
Weitzenböck formulae for L_f.
2024. "Linear stability of compact SRS" (Cao, Zhu)
gauge fixing; N_f.