Feb 19

① $Sym^2_I = \{h \mid h(J, J) = h\}$
$Sym^2_A = \{h \mid h(J, J) = -h\}$

$$Sym^2 = \mathbb{R}g \oplus \underbrace{Sym^2_{0,I} \oplus Sym^2_{0,A}}_{\text{traceless}}$$

② On $\Lambda^2 \otimes \Lambda^2$ $(S \cdot T)_{ijkl} = S_{ijpq} T_{pqkl}$
$SO(n) \rightsquigarrow (S \# T)_{ijkl}$

$tr_g : \Lambda^2 \otimes \Lambda^2 \longrightarrow Sym^2$
$tr_g(S) = \frac{1}{2}(S_{ipkp} + S_{kpip})$
Restricts to iso $\Lambda^-_g \oplus \Lambda^+_g \rightarrow Sym^2_0$.

Inverse $\mathring{h} \mapsto \frac{1}{2}\pi_{-,+}(\mathring{h} \circledast g)$
$\pi_{-,+} : \Lambda^- \otimes \Lambda^+ \longrightarrow \Lambda^2 \otimes \Lambda^2$

$\alpha, \beta \in \Lambda^2$ $\alpha \circ \beta = tr_g(\alpha \otimes \beta)$

③ $\delta_{f,a} = e^{(1-a)f} \delta^L e^{(a-1)f}$

$\rho \in \mathcal{H}^{1,1}_f$ $\rho = Ric(J-, -)$
On a Kähler Ricci soliton $Ric + \nabla^2 f = \frac{\lambda}{2} g$
In general $L_f h = (\Delta_f u) g + 2u Ric$
$+ tr\left(\Delta^L_{H,f,\circ} S + S \circ \left(W^+ + \left(\frac{\lambda + scal}{3}\right) Id_A\right)\right)$

Formula simplifies on Sym^2_I, Sym^2_A.

Lemma. (M, g, J, f) Ricci shrinker (Kähler)
If $\dim \mathcal{H}_f^{1,1} \geq 2$, then (M, g, J, f) is linearly unstable.

(To prove BCCD unstable)

Suppose $S_I = \gamma \otimes \omega$, ω Kähler form, $\gamma \in \Lambda^2$.

$$L_f h_I = ((\Delta^2_{H,f} + \lambda)\gamma)\omega$$

Let (M, g, J, f) a KRS. Assume $\dim \mathcal{H}_f^{1,1} \geq 2$.
Note $\rho \in \mathcal{H}_f^{1,1}$, there is $\gamma \in \mathcal{H}_f^{1,1}$ s.t. $\gamma \perp_{L^2_f} \rho$

Let $h = -h_I = tr(\gamma \otimes \omega)$, then $L_f h = \lambda \gamma \circ \omega = \lambda h$
Shrinker $\langle L_f h, h \rangle = \lambda |h|^2 > 0$

$t \mapsto e^t h$. $\quad (\partial_t - L_f) e^t h = 0 \quad \|e^t h\| < \infty$
$t \in [-\infty, 0]$ ancient solution.

BCCD $(M^4, J) = (Bl_q(\mathbb{C} \times \mathbb{CP}^1), J)$
Blow up at $p_0 = (0, p)$. Let $q \in \mathbb{CP}^1 \setminus \{p\}$

$$\pi : M \to \mathbb{C} \times \mathbb{CP}^1$$

$\Sigma_1 = \pi^{-1}(p_0)$, $\quad \Sigma_2 = \pi^{-1}(\{1\} \times \mathbb{CP}^1)$.

$\tilde{\Sigma} = \overline{\pi^{-1}(\{0\} \times \mathbb{CP}^1 \setminus \{p\})}$, $\quad \mathbb{C}_q = \pi^{-1}(\mathbb{C} \times \{q\})$.

Lemma. There are two (1,1)-forms $\tilde{\pi}_1, \tilde{\pi}_2$ compactly supported and disjoint supp.

$$\int_{\tilde{\Sigma}} \tilde{\gamma}_1 = 1 \qquad \int_{\mathbb{C}_1} \tilde{\gamma}_2 = 0$$

$[\tilde{\gamma}_i] = \{\tilde{\gamma}_i + d\eta\}$ $\quad \delta_{f,0}\gamma_i = 0 = \delta_{f,0}\tilde{\gamma} + \delta_{f,0} d\eta$

Rmk $\quad \Delta_{H,f,0} d\eta = -\Delta_{H,f,0}\tilde{\gamma}_i$ are (1,1)-forms because Δ preserves (1,1)-forms.

WLOG. $\delta_{f,0}\eta = 0$, we want to solve η:

$$\begin{cases} \Delta_{H,f,0}\eta = -\delta_{f,0}\tilde{\gamma}_i \\ \delta_{f,0}\eta = 0 \end{cases}$$

Claim: solution exists.

$$d\delta_{f,0} + \delta_{f,0} d = -\Delta_{f,0} + Ric_f = -\Delta_f + \frac{1}{2}$$