Feb 26 Thm 3 (M^4, g, J, f) $Ric_g + Hess f = \frac{1}{2} g$.

L_f = Lichnerowicz Laplacian, acting on $\Lambda^2 \otimes \Lambda^2$ and Sym^2.

If $\lambda \le 0$, then $\forall h \in H_f^2$, $\langle L_f h, h \rangle = 0$

i.e. (M^4, g, J, f) is stable.

$h = ug + tr(S) \longrightarrow h = tr(S)$

pf. Outline

① Weitzenböck formula of L_f. (Prop 8)

② Action of self dual curvature $W^+ + (\lambda - \frac{scal}{?}) Id_{\Lambda_g^2}$

③ Eqn (6) $tr(\Delta^L_{H,f,\circ} S_I)$ ← J-inv. symm. 2-tensor

④ Formulae (9) (10) for $h = h_I + h_A$.

$L_f h_I$, $L_f h_A$

⑤ Inequalities (11) (12) for $\langle L_f h_I, h_I \rangle$, $\langle L_f h_A, h_A \rangle$.

①. For $h = tr(S)$, $S \in \Lambda_g^- \otimes \Lambda_g^+$.

$L_f h = tr(\Delta^L_{H,f,\circ} S + S \circ \bar{R}^+)$

$\bar{R}^+ = W^+ + (\lambda - \frac{scal}{?}) Id_{\Lambda_g^2}$ self-dual

②. Action of $\bar{R}^+$. $W \in \Lambda^2 \otimes \Lambda^2 = End(\Lambda^2)$.

$\Lambda^2 = \Lambda_g^+ \oplus \Lambda_g^-$

$\Lambda^2 \otimes \Lambda^2 = (\Lambda_g^+ \oplus \Lambda_g^-) \otimes (\Lambda_g^+ \oplus \Lambda_g^-)$.

$= + \otimes + \quad + \otimes - \quad - \otimes + \quad - \otimes -$.

W in here.

So $W = W^+ + W^-$.

Split $\Lambda_g^+ = \langle \omega \rangle + \mathrm{Re}\,\Lambda^{2,0}$.

$\quad\quad$ ↖ Kähler form.

$$\Lambda^2 \otimes \mathbb{C} = \Lambda^{2,0} \oplus \Lambda^{1,1} \oplus \Lambda^{0,2}$$

$$\Lambda_g^- \cong \Lambda_0^{1,1} = \{ \beta \in \Lambda^{1,1} : \beta \wedge \omega = 0 \}.$$

Action of $\bar{R}^+$ on $\langle \omega \rangle$ and $\mathrm{Re}\,\Lambda^{2,0}$:

$$\bar{R}^+ \omega = \lambda \omega, \quad \bar{R}^+|_{\mathrm{Re}\Lambda^{2,0}} = \left(\lambda - \frac{scal}{3}\right) Id_{\Lambda^{2,0}}$$

<u>Prop 7</u> For $\phi \in \Lambda^2$, $\Delta \phi = \Delta_H \phi - \left(W - \frac{scal}{3} Id_{\Lambda^2}\right) \phi$.

plug in $\phi = \omega$:

$$\nabla^* \nabla \omega = \Delta_H \omega - \left(W - \frac{scal}{3} Id_{\Lambda^2}\right) \omega.$$

$$\begin{matrix} \| & & \| & \\ 0 & & 0 & \text{harmonic.} \end{matrix}$$

$\Rightarrow \left(W - \frac{scal}{3} Id_{\Lambda^2}\right) \omega = 0$. and $\omega \in \Lambda_g^+$

$\Rightarrow \bar{R}^+ \omega = \lambda \omega$.

$$W^+ = \begin{pmatrix} \frac{scal}{6} & & 0 \\ & -\frac{scal}{12} & \\ 0 & & -\frac{scal}{12} \end{pmatrix}$$

$Tr_{End\,\Lambda^2}(W) = 0$

$Tr_{End\,\Lambda^2}(W^+) = 0$.

For $\Lambda_g^+ = \langle \omega \rangle \oplus \mathrm{Re}\,\Lambda^{2,0}$

$$\bar{R}^+|_{\Lambda^{2,0}} = \left(\lambda - \frac{scal}{2}\right) Id_{\mathrm{Re}\Lambda^{2,0}}$$

③ Apply $\Delta^{L}_{H,f,\circ}(\gamma \otimes \phi) = (\Delta_{H,f,\circ}\gamma) \otimes \phi + 2\nabla\gamma \otimes \nabla_f \phi$

$+ \gamma\, \Delta_f \phi + R \# (\gamma \otimes \phi)$

with $\phi = \omega$. and use $\nabla \omega = 0$. $\Delta_f \omega = 0$

When $\gamma \in \Lambda_g^{-} = \Lambda_0^{1,1}$ $R \# (\gamma \otimes \omega) = 0$

$\Rightarrow \Delta^{L}_{H,f,\circ}(\gamma \otimes \omega) = (\Delta_{H,f,\circ}\gamma) \otimes \omega.$

④ Apply ① to $h = tr(S)$. $h_I = tr(S_I)$

$h_A = tr(S_A)$.