Mar 5

Bamler - Cifarelli - Conlon - Deruelle

BCCD soliton

Thm (+ [CCD 22]) classifies Kähler grad shrinkers in complex 2D. complete, bdd scalar

Thm (Li-Wang) complete grad shrinker Kähler $\longrightarrow$ bdd scalar.

i.e. such a shrinker is

compact (i) Kähler-Einstein on Fano ($c_1(M)>0$) or compact shrinker Wang-Zhu.

noncompact $\Rightarrow$ connected to ∞.

Kähler form ω. locally $i\partial\bar\partial\varphi$. $\varphi\in C^\infty(M;\mathbb{R})$.

Ricci form $Ric(\omega)(\cdot,\cdot) = Ric(g)(J\cdot,\cdot)$

$= i\partial\bar\partial \log\det g$

Shrinking gradient Kähler Ricci soliton

$(M^{2n}, g, J, \nabla f)$ holomorphic

$$Ric(g) + \nabla^2 f = g.$$

- $R(g)\to 0$ along all integral curves of ∇f

(ii) $\mathbb{C}^2$ flat Gaussian shrinker $\nabla f = r\partial r$.

(iii) "blowdown"/FIK shrinker.

$U(2)$ invariant on $Bl_0(\mathbb{C}^2)$.

• $R(g) \not\to 0$ along some ∇f trajectory: (iv) $\mathbb{C} \times \mathbb{P}^1$.
(v) [CCD 22] a toric Kähler shrinker on $Bl_x(\mathbb{C} \times \mathbb{P}^1)$
(if it exists, unique)
$\longrightarrow$ Thm A

$$\partial_t R = \Delta R + 2|Ric|^2 \geq \frac{2}{n} R^2 + \Delta R.$$
$$R \sim \frac{1}{t} \qquad (-\infty, 0)$$
$\Big\}$ $R \geq 0$ for shrinker.

Kähler - Ricci flow (M^{2n}, ω, J)

$$\partial_t \omega = - Ric(\omega)$$

$(M^{2n}, \omega(t))$ remains Kähler w.r.t the fixed J.

since $c_1(M) \cdot 2\pi = [-Ric(\omega)]$

$\Rightarrow [\omega(t)] = [\omega_0] - 2\pi c_1(M) t. \in$ "Kähler cone"

$C_M = \{[\alpha] \mid$ admits Kähler form w.r.t. $J\}$ $\nearrow$

<u>Thm</u> if $[\omega_0] - 2\pi c_1(M) t \in C_M$ then KRF exists up to t.

Finite time singularities. $T < \infty$ on $[0, T)$

• $\sup (T-t) |Rm| < \infty$ Type I (*)
• $\sup (T-t) |Rm| = \infty$ Type II

Enders-Müller-Topping (*) $\longrightarrow$ rescale $\longrightarrow$ grad. shrinker.

Outline of the pf of Thm A $\quad = N.$

1. toric KRF on $Bl_x(\mathbb{P}^1\times\mathbb{P}^1)$ that contract ~~exp~~ exceptional divisor at $T<\infty$. and volume control.
2. Bamler, analyze blowup. exclude orbifold sing. Show the sing is Type I!
3. $vol(N) \to 0$ as $t \to T$. rule out FIK by [CCD22] $\Rightarrow$ (N) or (V). volume computation $\longrightarrow$ not $\mathbb{C}\times\mathbb{P}^1$.

1. initial metric $S^1\times S^1$ on $\mathbb{P}^1\times\mathbb{P}^1$.
p, q fixed point of $S^1 \circlearrowright \mathbb{P}^1$, let $x=(p,p)$.
Take $Bl_x(\mathbb{P}^1\times\mathbb{P}^1) = N$

$$\begin{tikzcd} Bl_x(\mathbb{P}^1\times\mathbb{P}^1) \arrow[d, "\sigma"] \\ \mathbb{P}^1\times\mathbb{P}^1 \end{tikzcd}$$

E exceptional divisor.

$D_1 = \sigma^{-1}(\mathbb{P}^1\times\{q\})$, $D_2 = \sigma^{-1}(\{p\}\times\mathbb{P}^1)$.
intersection form $[D_i]^2 = 0$, $[D_1][D_2] = 1$
$[D_1][E] = 0$ $\quad [E]^2 = -1$.

<u>Lemma</u>. can choose $g(0)$, Kähler on N s.t.
$vol(D_1, g(t)) \to 0$ $\quad vol(D_2, g(t)) \not\to 0$
E $\quad vol(N, g(t))(T-t)^{-2} \not\to 0$
N

2. by contradiction. if not $\exists (x_i, t_i)$ s.t.

$$|Rm(x_i)|(T-t_i) \to \infty.$$

Bamler's compactness theory $\Rightarrow$

$$(N, (T-t_i)^{-1} g((T-t_i)t + t_i), \nu_{x_i, t}) \xrightarrow{\mathbb{F}\text{-conv}} (\mathcal{X}, (\nu_{\infty, t})_{t \le 0})$$

ancient metric flow x_∞ pt.

smooth away from codim 4 set and singular at x_∞

$\longrightarrow$ isolated orbifold sing. $\mathbb{R}^4/\Gamma$.

non flat, flow of metric soliton

$(N, g(t))$ type I sing as $t \to T$.

not compact

$R > 0$

vol > 0 so not FIK

[CCD22] $\Rightarrow$ $\mathbb{C} \times \mathbb{P}^1$ or (v).