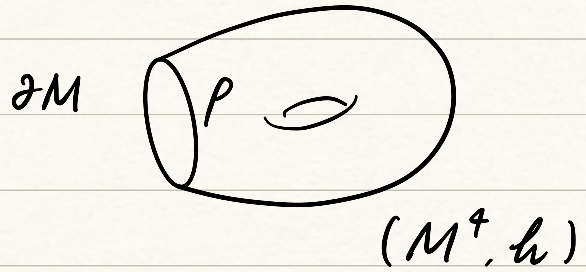
▹ Conformal compactification



$Ric = -3h$ Einstein cond.

$(\bar{M}, g = \rho^2 h)$ conf. compactification

$[g|_{\partial M}]$ conf. inf.

$|d\rho|_g^2 = 1$ asy. hyperbolic

Geodesic bdf. r, $g' = r^2 h$ s.t.

$|\nabla_{g'} r|_{g'} = 1$ near ∂M

▹ CCE fill-in problem Given $[\hat{g}]$ on ∂M,
find h on M s.t. $[\rho^2 h] = [\hat{g}]$

BVP
$$\begin{cases} Ric_h = -3h & \text{on } M \\ [\rho^2 h|_{\partial M}] = [\hat{g}] & \text{on } \partial M \end{cases}$$

▷ Fefferman - Graham expansion

$$g' = dr^2 + \hat{g} + r^2 g_{(2)} + r^3 g_{(3)} + \cdots$$

locally determined by $\hat{g}$ global depend on h and $\hat{g}$

h is determined by $\hat{g}$ and $g_{(3)}$, up to isometry

- locally a fill-in is possible
  real-analytic bdy. LeBrun, Fefferman - Graham
  smooth boundary. Gursky - Szekelyhidi

- global
  1. perturbation approach: Graham - Lee, Lee, Anderson
  2. 4D. classification of local deformation of self-dual PE metric near hyperbolic B^4. Biquard
  3. compactness result, perturbative existence
     Chang. Ge. Jin. Qing

Goal: Construct PE metric using Kähler reduction

$(\Sigma_g, g^\#)$ closed, oriented, smooth

genus ≥ 1

$$|k| \cdot \left(2\sqrt{6} \max |W_h^+|_h\right)^{1/3} = \frac{1}{2}$$

$k \in \mathbb{R} \cup \{\pm\infty\}$

orbit length

tuple $(\deg, \chi, k, \underline{a}, \underline{p})$

area

degree of P Euler characteristic $\chi = 2 - 2g$

Thm 1.3. (conformally Kähler PE)
$(\Sigma_g, g^\#)$ closed, oriented, smooth $g \geq 1$
$(\deg, \chi, k, \underline{a}, \underline{p})$ admissible Def 3.7, 3.8 below

$\Rightarrow$ $\exists$ PE (M, h) with conf. inf $[g^b]$ implicitly determined by $g^\#$ and the tuple where

- $M \underset{\text{diffeo.}}{\approx}$ $\begin{array}{c} P^{\deg} \\ \downarrow \\ \Sigma_g \end{array}$

- $(\partial M, g^b)$ carries a free isometric S^1-action with quotient $(\Sigma_g, g^\#)$ and const. $\underline{p}$

Special case: ASD

anti self-dual $\iff$ $k = \pm\infty$

admissible : $deg < -\chi/2$

$\underline{p} = \pi$

$\underline{a} = -\pi\,(deg + \chi/2)$

Definition 3.7. A five tuple $(deg, \chi, k, \mathfrak{a}, \mathfrak{p})$, where $deg \in \mathbb{Z}$ and $\chi = 2 - 2\,\mathsf{g}$ with $\mathsf{g} \in \mathbb{Z}_{>0}$, is said to be admissible if one of the following holds:

- $deg \neq -\chi$, $k \neq \frac{1}{\sqrt[3]{48}}$, and
$$\mathfrak{p} = \frac{96\pi k^3}{96k^3+1}, \qquad \mathfrak{a} = -\frac{\pi(\chi + deg)}{2\left(1 - \frac{1}{48k^3}\right)} - \frac{\pi\, deg}{2 + \frac{1}{48k^3}}.$$
Moreover, either
$$k \in \left[-\infty, -\tfrac{1}{\sqrt[3]{96}}\right) \cup \left(\tfrac{1}{\sqrt[3]{48}}, \infty\right], \qquad deg < -\chi\frac{96k^3+1}{192k^3-1},$$
or
$$k \in \left(\tfrac{1}{\sqrt[3]{192}}, \tfrac{1}{\sqrt[3]{48}}\right), \qquad deg > -\chi\frac{96k^3+1}{192k^3-1}.$$
- $deg = -\chi$, $k = \frac{1}{\sqrt[3]{48}}$, and
$$\mathfrak{p} = \tfrac{2\pi}{3}, \qquad \mathfrak{a} > 0 \text{ arbitrary.}$$

Definition 3.8. A five tuple $(deg, \chi, k, \mathfrak{a}, \mathfrak{p})$, where $deg \in \mathbb{Z}$ and $\chi = 2$, is said to be admissible if one of the following holds:

- $deg \neq -\chi$, $k \neq \frac{1}{\sqrt[3]{48}}$, and
$$\mathfrak{p} = \frac{96\pi k^3}{96k^3+1}, \qquad \mathfrak{a} = -\frac{\pi(\chi + deg)}{2\left(1 - \frac{1}{48k^3}\right)} - \frac{\pi\, deg}{2 + \frac{1}{48k^3}}.$$
Moreover, either
$$k \in \left[-\infty, -\tfrac{1}{\sqrt[3]{96}}\right) \cup \left(\tfrac{1}{\sqrt[3]{48}}, \infty\right], \qquad deg < -\chi,$$
or
$$k \in \left[\tfrac{1}{\sqrt[3]{192}}, \tfrac{1}{\sqrt[3]{48}}\right), \qquad deg > -\chi,$$
or
$$k \in \left(0, \tfrac{1}{\sqrt[3]{192}}\right), \qquad -\chi < deg < -\chi\frac{96k^3+1}{192k^3-1}.$$
- $deg = -\chi$, $k = \frac{1}{\sqrt[3]{48}}$, and
$$\mathfrak{p} = \tfrac{2\pi}{3}, \qquad \mathfrak{a} > \tfrac{\pi\chi}{3} \text{ arbitrary.}$$

- Toda type eqn.

$$
\begin{cases}
(1) & (e^w)_{\xi\xi} + \Delta_\Sigma w - 2K_\Sigma = -\xi e^w \dfrac{12 - 6\xi\partial_\xi w}{12k^3 + \xi^3} \\[2ex]
(2) & W = \dfrac{12 - 6\xi\partial_\xi w}{12 + \xi^3/k^3}
\end{cases}
$$

$$|k| \cdot \left(2\sqrt{6} \max |W_h^+|_h\right)^{1/3} = \frac{1}{2}$$

Thm 1.4 (cohom 1 PE metric)

On B^4, or $M \approx$ complex line bundle $\begin{array}{c} P \\ \downarrow \\ \Sigma \end{array}$

$\exists$ a 1-param. family of cohomogeneity 1 PE metrics with regular conf. Kähler geometry.

Thm 1.5. (Dirichlet BVP)

$\forall\, \varphi \in C^{2,\alpha}(\Sigma)$, $\int_\Sigma e^\varphi \, dvol_\Sigma = \underline{a}$, $\exists!$ sol. w of

$$
\begin{cases}
(1) \\
w|_{\xi=0} = \varphi
\end{cases}
$$

s.t. $w - \log\left(\frac{1}{2} - \xi\right) \in C^{2,\alpha}([0, \tfrac{1}{2}] \times \Sigma)$

$C^\infty((0, \tfrac{1}{2}] \times \Sigma)$

4D $\Lambda^2 T^* M = \Lambda^+ \oplus \Lambda^-$

$$Rm = \begin{pmatrix} W^+ + \frac{s_h}{12} Id & \mathring{Ric} \\ \mathring{Ric} & W^- + \frac{s_h}{12} Id \end{pmatrix}$$

▷ Derdzinski's Classification of Einstein h

Type I ASD. $W^+ = 0$

Type II W^+ has two distinct evalue ⟶
if h locally conf to Kähler g, conf. factor determined by $|W_h^+|_h$ $g = (2\sqrt{6}\,|W_h^+|_h)^{2/3} h$

Type III ——— three ———
if h locally conf to Kähler g, ruled out by

Kähler form ω, Kähler metric g
$\Lambda^+ = \mathbb{R}\omega + Re\,\Lambda^{2,0}$

$$W_g^+ = \begin{pmatrix} s_g/6 & & \\ & -s_g/12 & \\ & & -s_g/12 \end{pmatrix}$$

Focusing on PE + conf. to Kähler g
Lem 2.2. (sign of s_g)
$s_g > 0$ or < 0 or $\equiv 0$ on M

▷ conf. Kähler PE.

- Killing v.f. K on (M, h) extends to any conf. comp. as a conf. Killing field tangent to ∂M.

- Kähler conf change g yields a compactification $\iff K \not\equiv 0$ on ∂M

▷ Regular conformally Kähler geometry on (M, h)

$K|_{\partial M}$ periodic, integrates to a free S^1-action so that

$$\begin{array}{ccc} S^1 & - & \partial M \\ & & \downarrow \\ & & \Sigma \end{array}$$

$S^1 \circlearrowright M$ is Hamiltonian w.r.t. ω.

$\longrightarrow \xi$. moment map.

▷ Kähler reduction away from $\mathrm{crit}(\xi)$.

$$g = W\, d\xi^2 + \frac{1}{W}\eta^2 + W e^{w} g_\Sigma$$

η: conn. 1-form dual of K.

W, e^{w}: functions of ξ

g_Σ: const. curvature $K_\Sigma = 0, \pm 1$

$$g = \xi^2 h, \quad K = J\nabla_g \xi, \quad \xi : \bar{M} \longrightarrow [0, \tfrac{1}{2}]$$

ξ is Morse-Bott

Type I. ASD

Prop 2.3. h Einstein, ASD, $s_h \not\equiv 0$

Rmk 2.5. not necessarily global

$K \neq 0 \iff$ h locally conf. to Kähler g w/ $s_g = 0$ on an open set

Toda

$$g = \zeta^2 h = W(d\zeta^2 + e^v(dx^2+dy^2)) + W^{-1}\eta^2, \tag{5}$$

$$(e^v)_{\xi\xi} + v_{xx} + v_{yy} = 0, \tag{6}$$

$$W = 1 - \frac{1}{2}\zeta v_\zeta, \tag{7}$$

$$d\eta = (We^v)_\zeta dx dy + W_x dy d\zeta + W_y d\zeta dx. \tag{8}$$

$K = J\nabla_g \zeta$, $\quad J dx = dy$, $\quad J d\zeta = \frac{1}{W}\eta$

ζ not necessarily a bdf. need $K \neq 0$ on ∂M

Prop 2.11 $K \not\equiv 0 \iff \zeta = rf$, $f \in C^\infty(M)$, $f > 0$ a smooth bdf.

pf uses FG expansion

Example hyperbolic B^4 p.7

Type II. PE, simply conn $\quad \lambda = 2\sqrt{6}\,|W_h^+|_h \neq 0$

extremal Kähler $\quad g = \lambda^{2/3} h$

$|s_g| = \lambda^{1/3}$ $\quad \to \quad s_g > 0$ or < 0 on M

$\to$ Killing v.f. $K = J \nabla_g s_g$

(10) $$g = \zeta^2 h = W(d\zeta^2 + e^v(dx^2 + dy^2)) + W^{-1}\eta^2.$$

Here, $\zeta := s_g$ is the moment map for $\mathcal{K}$, $x + iy$ is a holomorphic coordinate we pick on the Kähler reduction. Under (10), the Type II Einstein equation reduces to the following equations

(11) $$(e^v)_{\zeta\zeta} + v_{xx} + v_{yy} = -\zeta W e^v,$$ twisted $SU(\infty)$ Toda eqn

(12) $$W = \frac{12 - 6\zeta v_\zeta}{12 + \zeta^3},$$

(13) $$d\eta = (We^v)_\zeta dxdy + W_x dyd\zeta + W_y d\zeta dx.$$

$K \neq 0$ on ∂M $\Rightarrow$ g is a conf. comp.
(by Prop 2.12)

Example AdS - Schwarzschild p. 9.

Canonical conf. change $\quad g^\# = k^2 g = \overbrace{(k\zeta)}^{\xi}{}^2 h$

$$k \cdot \left(2\sqrt{6} \max |W_h^+|_h\right)^{1/3} = \frac{1}{2}\, \mathrm{sgn}(s_g)$$

so that $\xi \in [0, \frac{1}{2}]$

Write $\xi := k\zeta, v_\sharp := v + \log k^2, \eta_\sharp = k\eta$. The Ansätze (10)-(13) over the end of (M, h) now become

$$g_\sharp = W(d\xi^2 + e^{v_\sharp}(dx^2 + dy^2)) + W^{-1}\eta_\sharp^2, \tag{19}$$

$$e^{v_\sharp}_{\xi\xi} + (v_\sharp)_{xx} + (v_\sharp)_{yy} = -\xi e^{v_\sharp} \frac{12 - 6\xi\partial_\xi v_\sharp}{12k^3 + \xi^3}, \tag{20}$$

$$W = \frac{12 - 6\xi\partial_\xi v_\sharp}{12 + \xi^3/k^3}, \tag{21}$$

$$d\eta_\sharp = (We^{v_\sharp})_\xi dx dy + W_x dy d\xi + W_y d\xi dx. \tag{22}$$

We can further simplify by writing $e^{v_\sharp}(dx^2 + dy^2)$ as $e^{w_\sharp} g_\Sigma$. Then the Ansätze (19)-(22) become

$$g_\sharp = W(d\xi^2 + e^{w_\sharp} g_\Sigma) + W^{-1}\eta_\sharp^2, \tag{ansatz-1$_k$}$$

$$e^{w_\sharp}_{\xi\xi} + \Delta_\Sigma w_\sharp - 2K_\Sigma = -\xi e^{w_\sharp} \frac{12 - 6\xi\partial_\xi w_\sharp}{12k^3 + \xi^3}, \tag{ansatz-2$_k$}$$

$$W = \frac{12 - 6\xi\partial_\xi w_\sharp}{12 + \xi^3/k^3}, \tag{ansatz-3$_k$}$$

$$d\eta_\sharp = \star((d + \partial_\xi w_\sharp d\xi)W). \tag{ansatz-4$_k$}$$

canonical infinity $\quad g^\flat = g|_{\partial M}$

2D infinity $\quad g^\natural = g_\Sigma$

ξ is Morse-Bott $\qquad \mathrm{Hess}_g \xi$ non-degenerate

$\mathrm{crit}(\xi) = \bigsqcup_j C_j$

compact submanifolds

$C_j = \{pt\}$ or Σ^2.