PE $\quad (M^4, h)$ conformal to Kähler

g Kähler scalar flat

Ch 3 of LL25

Decoupled solution

$\quad \mathrm{crit}(\xi) = \{pt\}$ or Σ

$$M = B^4 \quad \text{or} \quad L \to \Sigma$$

at non critical value x_0.

$$\begin{array}{ccc} S^1 & \text{---} & \xi^{-1}(x_0) \\ & & \downarrow \\ & & \Sigma \end{array}$$

Kähler reduction

p = period of K.

$W \equiv 1$ over ∂M so p = length of S^1-action

$$\frac{1}{p}\int_\Sigma d\eta = \deg, \quad \text{degree of } S^1\text{-bundle}$$

degree of $\mathbb{C}$-bundle

$$\begin{cases} (13) \quad d\eta = (We^{u})_\xi \underbrace{dx\,dy}_{dvol_\Sigma} + W_x\, dy\, d\xi + W_y\, d\xi\, dx \\ \\ \frac{1}{p}\int_\Sigma d\eta = \deg \end{cases}$$

$$\Rightarrow \quad (25) \quad \int_\Sigma We^{w}\, dvol_\Sigma = (\deg\cdot p)\,\xi + a$$

evaluate at $\xi = 0$,

$W = 1 \Rightarrow a = \mathrm{Area}(\Sigma)$

Take ansatz -2h

integrate w.r.t. Σ

$$(e^w)_{\xi\xi} + \underbrace{\Delta_\Sigma w}_{0} - \underbrace{2K_\Sigma}_{4\pi\chi(\Sigma)} = -\xi \underbrace{W e^w}_{(\deg\cdot p)\xi + a} \frac{1}{k^3}$$

use (25)

integrate w.r.t. ξ twice $\Rightarrow$

(26) $$\int_\Sigma e^w \, dvol_\Sigma = -\frac{\deg\cdot p}{k^3}\xi^4 - \frac{a}{6k^3}\xi^3 + 2\pi\chi(\Sigma)\xi^2 + 2(\deg\cdot p)\xi + a$$

Ch 4 Dirichlet BVP

$(\Sigma, g^{\#})$ of genus $g \geq 1$

M = underlying space of $\begin{array}{c} \mathbb{C} - L \\ \downarrow \\ \Sigma \end{array}$

$(\deg, \chi, k, a, p)$ admissible

Filling in with

(M, h, g) regular conf. Kähler PE metric

s.t. 2D infinity $h = g^{\#} = e^{\varphi} \underbrace{g_\Sigma}_{\text{const. } K_\Sigma}$

p = length of S^1-fibers of $g^{\flat}$

a = area of $(\Sigma, g^{\#})$

k s.t. $k = \pm\infty$ if h ASD, otherwise

$$k \cdot \left(2\sqrt{6} \max |W_h|_h^+\right)^{1/3} = \frac{1}{2} \operatorname{sgn}(s_g)$$

Kähler reduction

BVP
$$\begin{cases} (e^w)_{\xi\xi} + \Delta_\Sigma w - 2K_\Sigma = -\xi e^w \dfrac{12 - 6\xi \partial_\xi w}{12k^3 + \xi^3} \\ w|_{\xi=0} = \varphi, \quad \int_\Sigma e^{\varphi} \, d\mathrm{vol}_\Sigma = a \end{cases}$$

$$w - \log\left(\frac{1}{2} - \xi\right) \in C^{2,\alpha}\left(\left[0, \tfrac{1}{2}\right] \times \Sigma\right)$$

Normalization $\bar{w}(\xi) = \log\left(\frac{1}{\mathrm{Vol}_\Sigma}\int_{\{\xi\}\times\Sigma} e^w \, d\mathrm{vol}_\Sigma\right)$

$e^{\bar{w}}$ polynomial in ξ, >0 in $[0,\frac{1}{2})$, has a simple zero at $\xi = \frac{1}{2}$.

$$(e^{\bar{w}})_{\xi\xi} + \xi e^{\bar{w}} \frac{12 - 6\xi\partial_\xi \bar{w}}{12k^3+\xi^3} = 2K_\Sigma$$

Set $u = w - \bar{w}$ $\quad a(\xi) = \frac{-6\xi^2}{12k^3+\xi^3}$ $\quad b(\xi) = \frac{12\xi}{12k^3+\xi^3}$

$\psi(\xi) = e^{\bar{w}}/(1-\xi)$

Then u :
$$\begin{cases} \Delta_\Sigma u + e^{\bar{w}}(e^u)_{\xi\xi} + 2(e^u)_\xi (e^{\bar{w}})_\xi \\ \qquad\qquad + a(\xi) e^{\bar{w}}(e^u)_\xi + 2K_\Sigma(e^u - 1) = 0 \\ u|_{\xi=0} = \varphi - \bar{w}|_{\xi=0} \end{cases}$$

C^0-bound of sol. by boundary data $\quad g \geq 1$

<u>Lem 4.4</u> Assume u is a $C^2([0,\frac{1}{2}]\times\Sigma)$ sol. Then

$$\sup_{[0,\frac{1}{2}]\times\Sigma} u \leq \sup_{\{0\}\times\Sigma} \underbrace{u^+}_{\max\{u,0\}}, \quad \inf_{[0,\frac{1}{2}]\times\Sigma} u \geq \inf_{\{0\}\times\Sigma} -\underbrace{u^-}_{\max\{-u,0\}}$$

pf. use max principle of Δ_Σ

Higher order estimate $g \geq 1$

<u>Thm 4.5</u> Assume u is a $C^2([0,\frac{1}{2}]\times\Sigma)$ sol.
$m \geq 1$, $0<\alpha<1$. If $u|_{\{0\}\times\Sigma} \in C^{m,\alpha}(\{0\}\times\Sigma)$

- Then $\exists\ C = C(m,\alpha, |u|_{C^{m,\alpha}(\{0\}\times\Sigma)})$ s.t.

$$|u|_{C^{m,\alpha}([0,\frac{1}{2}]\times\Sigma)} \leq C.$$

- $\forall\, n \in \mathbb{N}$, $\exists\ C_n = C(n, |u|_{C^0(\{0\}\times\Sigma)})$

$$|u|_{C^n([\frac{1}{4},\frac{1}{2}]\times\Sigma)} \leq C_n$$

pf. Step 1. Lift to $\mathbb{R}^4\times\Sigma$
Let $\vec{z} = (z_1, z_2, z_3, z_4) \in \mathbb{R}^4$, $\zeta = \frac{1}{2} - \frac{1}{4}|\vec{z}|^2$

lift func. $\vec{u}(\vec{z},x) = u(\frac{1}{2}-\frac{1}{4}|\vec{z}|^2, x)$
$(\vec{z},x) \in \Omega = \{|\vec{z}| < \sqrt{2}\}\times\Sigma$

corresponding PDE on Ω.

$$\Delta_x \vec{u} + \psi\Delta_{\vec{z}} e^{\vec{u}} - \frac{1}{2}(2\psi_\zeta + a\psi)e^{\vec{u}}\vec{z}\cdot\nabla_{\vec{z}}\vec{u} + 2K_\Sigma(e^{\vec{u}}-1) = 0$$

Lem 4.4 holds for $\vec{u}$. $\{\zeta = 0\}\times\Sigma$ corresp. $\Sigma\times S^3(\sqrt{2}) \subseteq \partial\Omega$

and $\quad c_1 |u|_{C^{m,\alpha}(\partial\Omega)} \leq |\vec{u}|_{C^{m,\alpha}(\partial\Omega)} \leq c_2 |u|_{C^{m,\alpha}(\partial\Omega)}$

Step 2. Rewrite the PDE in divergence form and apply De Giorgi-Nash-Moser interior and bdry est.

$$|\vec{w}|_{C^{1,\alpha}(\bar{\Omega}')} \leq C(1+\|\vec{u}\|_{C^0(\bar{\Omega})}) \leq C$$

$$|\vec{w}|_{C^{1,\alpha}(\bar{\Omega})} \leq C(1+\|\vec{u}\|_{C^0(\bar{\Omega})}+|\vec{u}|_{C^{1,\alpha}(\partial\Omega)}) \leq C$$

Schauder estimate + bootstrap

$\rightsquigarrow$ global $C^{m,\alpha}(\bar{\Omega})$ estimate

Step 3. Reduction to u

Lemma 4.6. *Let $f : [0, R^2] \to \mathbb{R}$ and define $F : \overline{B_R(0)} \subset \mathbb{R}^n \to \mathbb{R}$ by $F(v) = f(|v|^2)$. Assume $F \in C^{2m}(\overline{B_R(0)})$ for some $m \in \mathbb{N}$. Then $f \in C^m([0, R^2])$, and $|f|_{C^m([0,R^2])} \leq C\,|F|_{C^{2m}(\overline{B_R(0)})}$.*

Proof. Assume $m \geq 1$. Fix any $e \in S^{n-1}$ and define $g(t) := F(te) = f(t^2)$, which is an even C^{2m} function on $[-R, R]$. Thus, all odd derivatives of g vanish at 0. By subtracting the $2m$-th Taylor polynomial centered at 0, we may assume $g(0) = \cdots = g^{(2m)}(0) = 0$. Applying the integral remainder theorem to $g(t)$ gives $f(s) = \frac{1}{(2m-1)!}\int_0^{\sqrt{s}}(\sqrt{s}-\tau)^{2m-1}g^{(2m)}(\tau)\,d\tau$, with $s = t^2$. By induction on q, one shows that for any $1 \leq q \leq m$, $f^{(q)}(0) = 0$, $f^{(q)}(s) = \sum_{j=1}^{q} C_{q,j}\int_0^{\sqrt{s}} s^{\frac{j}{2}-q}(\sqrt{s}-\tau)^{2m-1-j}g^{(2m)}(\tau)\,d\tau$ when $s > 0$, $\lim_{s\to 0^+} f^{(q)}(s) = f^{(q)}(0)$, and

$$|f^{(q)}(s)| \leq \sum_{j=1}^{q}|C_{q,j}|\sqrt{s}\cdot s^{\frac{j}{2}-q}\cdot\sqrt{s}^{\,2m-1-j}\cdot\sup_{\tau\in[-R,R]}|g^{(2m)}(\tau)| = \sum_{j=1}^{q}|C_{q,j}|\,s^{m-q}\cdot\sup_{\tau\in[-R,R]}|g^{(2m)}(\tau)|.$$

The conclusion follows. □

integral reminder

2 derivatives of F $\sim$ 1 derivative of f

Existence and uniqueness

Lifted eqn.

$$\begin{cases} \Delta_x \vec{u} + \psi \Delta_{\vec{z}} e^{\vec{u}} - \frac{1}{2}(2\psi_\xi + a\psi) e^{\vec{u}} \vec{z} \cdot \nabla_{\vec{z}} \vec{u} + 2K_\Sigma (e^{\vec{u}} - 1) = 0 & \text{in } \Omega \\ \vec{u} = \vec{\varphi} & \text{on } \partial\Omega \end{cases}$$

Moduli space of sol. $\mathcal{M} = \{ \vec{u} \in C^{2,\alpha}(\overline{\Omega}) \mid \vec{u} \text{ sol.} \}$

Thm 4.8 $\mathcal{M}$ is a smooth Banach manifold and $\mathcal{M} \longrightarrow C^{2,\alpha}(\partial\Omega)$ is a diffeo.

$$\vec{u} \longrightarrow \vec{u}|_{\partial\Omega}$$

Proof. Define $\Phi : C^{2,\alpha}(\overline{\Omega}) \to C^{\alpha}(\overline{\Omega}) \times C^{2,\alpha}(\partial\Omega)$ by

$$\Phi(\mathbf{u}) = (\Delta_x \mathbf{u} + \psi \Delta_{\mathbf{z}} e^{\mathbf{u}} - \tfrac{1}{2}(2\psi_\xi + a\psi) e^{\mathbf{u}} \mathbf{z} \cdot \nabla_{\mathbf{z}} \mathbf{u} + 2K_\Sigma (e^{\mathbf{u}} - 1),\ \mathbf{u}|_{\partial\Omega}).$$

This is a smooth map between Banach spaces. Its derivative at $\mathbf{u}$ is $D\Phi_{\mathbf{u}}(\mathbf{v}) = (L_{\mathbf{u}}\mathbf{v},\ \mathbf{v}|_{\partial\Omega})$, where

(60) $$L_{\mathbf{u}}\mathbf{v} = \Delta_x \mathbf{v} + \psi \Delta_{\mathbf{z}}(e^{\mathbf{u}}\mathbf{v}) - \tfrac{1}{2}(2\psi_\xi + a\psi)\mathbf{z} \cdot \nabla_{\mathbf{z}}(e^{\mathbf{u}}\mathbf{v}) + 2K_\Sigma e^{\mathbf{u}}\mathbf{v}.$$

26

Step 1. Fredholm

$L_{\vec{u}}$ homotopic to $\Delta_x + \Delta_{\vec{z}}$

\+ index is invariant under cont. homotopy

Step 2. Invertibility of $D\Phi_{\vec{u}}$

surj $\Rightarrow \dim(\operatorname{coker} D\Phi_{\vec{u}}) = 0 \Rightarrow$ injectivity

ind $D\Phi_{\vec{u}} = 0$

So $D\Phi_{\vec{u}}$ is bij.

Schauder estimate ineq. $\longrightarrow$ bdd inverse

(∞-dim)

Surj use $\mathrm{im}\, T = (\ker T^*)^\perp$, and show trivial $\ker T^*$

$L_{\vec{u}}^*$ after simplification

$$L_{\vec{u}}^* \vec{w} = \Delta_x \vec{w} + e^{\vec{u}} \psi \Delta_{\vec{z}} \vec{w} + \frac{a\psi}{2} e^{\vec{u}} \vec{z} \cdot \nabla_{\vec{z}} \vec{w}$$
$$+ e^{\vec{u}} \psi \left(a + (\tfrac{1}{2} - \xi)(b - a_\xi)\right) \vec{w}$$

conf. related op. $L_{\vec{u},q}^* \vec{w} := e^{-q} L_{\vec{u}}^* (e^q \vec{w})$

$$q = \log \psi + \int_0^\xi a(\tau)\, d\tau \in C^2([0, \tfrac{1}{2}])$$

and apply the max. principle to $L_{\vec{u},q}^*$

$\Rightarrow$ Dirichlet BVP $\begin{cases} L_{\vec{u},q}^* \vec{w} = 0 \\ \vec{w}|_{\partial\Omega} = 0 \end{cases}$ admit only trivial sol.

Step 3.

$\mathcal{M}$ is a smooth Banach manifold since
$$\mathcal{M} = \Phi^{-1}\left(\{0\} \times C^{2,\alpha}(\partial\Omega)\right) \quad + \text{ implicit FT.}$$

$D\Phi_u$ is bij $\Rightarrow$ $\Phi|_{\mathcal{M}}$ is local diffeo.

Step 4. Proper boundary map

every bdd sequence of solution with converging boundary data admits a converging subseq.

$\Rightarrow$ $\mathcal{M} \longrightarrow C^{2,\alpha}(\partial\Omega)$ is a finite sheet covering

\+ local diffeo

max. principle

Step 5. Since $(\Phi|_{\mathcal{M}})^{-1}(0) = \{0\}$, 1-sheet covering.

$\Rightarrow$ global inj.