Einstein Filling with Prescribed Conformal Infinity

Xinran Yu

February 12, 2022

Abstract

Given a conformal class of metrics on the boundary of a manifold, one can ask for the existence of an Einstein metric whose conformal infinity satisfies the boundary condition.

In 1991, Graham and Lee studied this boundary problem on the hyperbolic ball. They proved the existence of metrics sufficiently close to the round metric on a sphere by constructing approximate solutions to a quasilinear elliptic system. In his monograph (2006), Lee discussed the boundary problem on a smooth, compact manifold-with-boundary. Using a similar construction, he proved the existence and regularity results for metrics sufficiently close to a given asymptotically hyperbolic Einstein metric. The proof is based on a linear theory for Laplacian and the inverse function theorem.

Contents

1 Introduction

These are the notes for a talk I gave in the graduate analysis seminar at UIUC. The purpose of this talk is to solve a boundary problem for Einstein metrics using the inverse function theorem. There won't be too much detail about the geometry, though I will make necessary comments to help audiences understand several essential notions in the theorem, such as Ricci curvature and boundary-defining function. Let's begin with setting up some notations and discussing some geometry backgrounds.

1.1 Notations

List of symbols encountered in this talk

1. $\overline{M}$: compact $(n+1)$-dim Riemannian manifold with boundary, $n \geq 3$.
2. M: interior of $\overline{M}$,
   g, h: Riemannian metric on M.

3. ∂M: boundary of $\overline{M}$,
   $\hat{g}, \hat{h}$ Riemannian metric on ∂M, which define conformal classes on ∂M.
4. ρ: smooth boundary defining function
   $\bar{g} = \rho^2 g$ Riemannian metric on M.

Remark 1.1.

- Conformal class: We call two metrics conformal if they differ by a smooth nonnegative function, for example g and $\rho^2 g$. This gives an equivalence relation, and we call $[g]$ the conformal class. We usually assume $\hat{g}$ of class $C^{l,\beta}$.
- Boundary defining function: ρ is a function defined on the manifold which vanishes to the first order. $\rho > 0$ on M, $\rho = 0$ on ∂M and $d\rho \neq 0$ on ∂M.
- The conformal class $[\hat{g}]$ is called the conformal infinity of the manifold.

1.2 Boundary problem

Let $\overline{M}$ be a manifold with boundary as above. Given a conformal infinity $\hat{g}$ at the boundary, we can ask if there is an Einstein metric g on M satisfying the boundary condition. That is

$$\mathrm{Ric}_g = -ng \qquad \text{(Einstein condition)}$$
$$g \text{ has the given conformal infinity} \qquad \text{(boundary condition)}$$

Here Ric_g stands for the Ricci curvature.

Remark 1.2 (Ricci curvature)**.** The Ricci curvature depends only on the Riemannian structure of the manifold. Once we fix a metric g on the manifold, it gives rise to a so-called Levi-Civita connection ∇_g. Intuitively the Levi-Civita connection allows us to compare local geometric objects, such as tangent vectors or tensors. We can define the Riemannian curvature tensor to measure the failure of recovering the original direction when parallel transporting a vector along a closed loop.

The Riemannian curvature is a (3,1)-tensor, which can be written explicitly using the metric (for our purpose, I'll avoid using Christoffel symbols, though let's use them here to save space)

$$R_{abc}{}^{d} = \frac{\partial \Gamma_{ac}{}^{d}}{\partial x^b} - \frac{\partial \Gamma_{bc}{}^{d}}{\partial x^a} + \sum_{e=1}^{n}(\Gamma_{ac}{}^{e}\Gamma_{be}{}^{d} - \Gamma_{bc}{}^{e}\Gamma_{ae}{}^{d}) = \Gamma_{ac,b}{}^{d} - \Gamma_{bc,a}{}^{d} + \Gamma_{ac}{}^{e}\Gamma_{be}{}^{d} - \Gamma_{bc}{}^{e}\Gamma_{ae}{}^{d},$$

where

$$\Gamma_{ab}{}^{c} = \frac{1}{2} g^{cd}(g_{da,b} + g_{db,a} - g_{ab,d}).$$

The Ricci curvature is given by taking the trace of the above tensor. One could think of taking traces as forcing two indices to be the same in the Riemannian curvature tensor. With the Einstein summation notation, we denote this by $\mathrm{Ric}_{ac} = R_{abc}{}^{c}$. The Ricci curvature is a $(2,0)$-tensor.

If g is a conformally compact metric, its Ricci curvature is of the following form

$$\begin{aligned} R_{jk} &= -\rho^{-2} n(\bar{g}^{il}\rho_i\rho_l\bar{g}_{jk}) + \rho^{-1}\mathcal{E}^1(\bar{g}) + \mathcal{E}^2(\bar{g}) \\ &= \rho^{-2}\mathcal{E}^0(\bar{g}) + \rho^{-1}\mathcal{E}^1(\bar{g}) + \mathcal{E}^2(\bar{g}), \end{aligned}$$

where $\mathcal{E}^0(\bar{g})$ denotes a polynomial in $\bar{g}$ and $\bar{g}^{-1}$; $\mathcal{E}^1(\bar{g})$ can contain the first derivatives of $\bar{g}$ and $\mathcal{E}^2(\bar{g})$ can contain the second derivatives of $\bar{g}$ or quadratic in first derivatives. Throughout this talk, I will refer to those notations frequently.

1.3 Main result

Theorem 1.3. *Given a manifold and a boundary defining function as above, and suppose there exists an Einstein metric h on the interior, which is conformally compact and of class $C^{l,\beta}$, with $2 \le l < n-1$ and $0 < \beta < 1$, having nonpositive sectional curvature. Denote $\hat{h} = \rho^2 h|_{\partial M}$. Then there exists $\epsilon > 0$ such that for any metric $\hat{g}$ on the boundary which is close to $\hat{h}$ (in the $C^{l,\beta}$ norm), there is an Einstein metric g on the interior which has $[\hat{g}]$ as its conformal infinity and of class $C^{l,\beta}$.*

Remark 1.4. In [1], Graham and Lee proved the above theorem on hyperbolic balls. To be more precise, given a conformal infinity that is sufficiently closed to the standard metric on the sphere, there is an Einstein metric that is closed to the hyperbolic metric.

In [2], Lee proved the above theorem for a more general compact manifold with boundary. The ideas of solving such problems are nearly identical. For the sake of concreteness, let's focus on the first case most of the time. And I'll point out the difference in handling B^{n+1} and a general manifold with a boundary when needed.

1.4 Outline of the proof

We will define an operator F which conveys the Einstein condition and perturb the operator by some $\Phi(g,t)$ (we will define it later) which depends on both the given metric and an auxiliary metric t. Note: t depends on the boundary condition.

Using the perturbed operator, we convert the original boundary value problem into solving the partial differential equation

$$Q(g,t) = F(g) - \Phi(g,t) = 0.$$

We will construct a sequence of asymptotically hyperbolic solutions to approximate the solution. The key theorem to obtain an exact solution is the inverse function (this requires some work). So it is important that the approximate solutions depend smoothly on the data. (We will define an extension operator in order to keep asymptotically hyperbolic condition.

2 Perturbation operator

We choose

$$F(g) = \mathrm{Ric}_g + ng \quad \text{and} \quad \Phi(g,t) = \delta_g^* g t^{-1} \delta_g G_g t,$$

so that the zero set of F is the collection of Einstein metrics.

Remark 2.1.

- G_g is the Einstein tensor / gravitational operator

- $\delta_g G_g \operatorname{Ric}_g = 0$ is the second Bianchi identity. This equation says the divergence of Einstein tensor is zero.

2.1 How is Φ chosen

The choice of Φ might be less obvious. We add this term on purpose so that the perturbed operator is elliptic.

As suggested in DeTurck's paper, the linearization of Ricci curvature fails to be elliptic because of its second term:

$$\operatorname{Ric}'_g(h) = \frac{d}{dt}\Big|_{t=0} \operatorname{Ric}(g+th) = \frac{1}{2}\Delta_L h - \delta^*(\delta G_g h).$$

However, the first term is the Lichnerowicz Laplacian, which we know is elliptic. This suggest we need to add a gauge-breaking term $\delta_g^* g t^{-1} \delta_g G_g t$ (this happens to be the divergence of harmonic map Laplacian of the identity $\mathrm{id} : (M,g) \to (M,t)$. One can check that Φ eliminates the second term to the second order.

2.2 Linearization of Q

In this section, let's discuss the linearization of the perturbed operator $Q(g,t)$, which will be used later on when we construct approximate solutions.

Proposition 2.2. *For metrics g, t and a symmetric 2-tensor r, we have*

$$\begin{aligned} D_1 Q_{(g,t)}(r) &= \frac{d}{ds}\Big|_{s=0} Q(g+sr,t) = D\operatorname{Ric}_g(r) + nr - D_1\phi_{(g,t)}(r) \\ &= \frac{1}{2}\Delta_g r + \mathcal{R}(r) + nr - \delta_g^*(\mathcal{C}(r) - \mathcal{D}(r)) - \mathcal{B}(r), \end{aligned}$$

where $\mathcal{R}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are defined in [1] Equation (2.5) and Lemma 2.3.

For a complete computation of the above formula, see [1] (Section 2). Here, let me emphasizes Q is elliptic, because its principal symbol is one-half the covariant Laplacian Δ_g.

If we further assume g and t are asymptotically hyperbolic metrics with $\bar{g}|_{\partial M} = \bar{t}|_{\partial M}$, there is a nice approximation for the Riemannian curvature

$$R_{ijkl} = -(g_{ik}g_{jl} - g_{il}g_{jk}) + O(\rho^3).$$

which implies $\mathcal{R}(r) + nr = -r + tr_g(r)g + O(\rho^{s+1})$, and the $\mathcal{B}, \mathcal{C}, \mathcal{D}$ part vanishes. Simplifying and decomposing the linearization of Q into trace part and trace-less part gives the following.

Proposition 2.3. *Suppose g, t are asymptotically hyperbolic metrics with $\bar{g}|_{\partial M} = \bar{t}|_{\partial M}$. Write $r = \rho^s \bar{q}$, where $\bar{q} \in C_2(\overline{M}, S^2)$. If $r = ug + r_0$ where r_0 is the trace-free, then*

$$\begin{aligned} D_1 Q_{(g,t)}(r) &= \frac{1}{2}\Delta_g r - r + tr_g(r)g + O(\rho^{s+1}) \\ &= \frac{1}{2}(\Delta_g + 2n)(ug) + \frac{1}{2}(\Delta_g - 2)(r_0) + O(\rho^{s+1}) \end{aligned}$$

Remark 2.4. The above linearization $D_1 Q$ with respect to g at a confromally compact Einstein metric h is

$$D_1 Q_{(h,h)} = \frac{1}{2}(\Delta_L + 2n),$$

which is an isomorphism between weighted Hölder spaces. That will be a crucial fact for us to construct approximate solutions.

[P194 Lemma 2.2 strictly negative Ricci curvature implies the solution is an Einstein metric.]

3 Functional spaces

In order to construct approximate solutions, let's first define the following function spaces:

- $C^{l,\beta}_{(0)}(\overline{M}; \Sigma^2 \overline{M})$ = the usual Banach space of functions on $\overline{M}$ with k-Hölder continuous derivatives

- $C^{l,\beta}_{(s)}(\overline{M};\Sigma^2\overline{M}) := \{u \in C^{l,\beta}_{(0)}(\overline{M};\Sigma^2\overline{M}) \mid u = O(\rho^s)\}$ for $0 \le s \le l+\beta$. This is called weighted Hölder space.

Remark 3.1. One needs to be more careful in dealing with function spaces on a manifold with boundary. See [2] Chapter 3 for a complete discussion about functional space and Sobolev embedding. In most cases, the properties of function spaces are proved by applying Taylor's expansion locally in the background charts.

We will apply the following lemma

Lemma 3.2.

(i) $C^{l,\beta}_{(s)}(\overline{M};\Sigma^2\overline{M}) = \left\{u \in C^{l,\beta}_{(0)}(\overline{M};\Sigma^2\overline{M}) : \left.\frac{\partial^i u}{\partial \rho^i}\right|_{\partial M} = 0, \forall i \in [0,s)\right\}.$

(ii) $\rho^{-j} C^{l,\beta}_{(s)}(\overline{M};\Sigma^2\overline{M}) \subset C^{l-j,\beta}_{(s-j)}(\overline{M};\Sigma^2\overline{M})$ *for* $0 < j \le s$.

Proof. Note that locally the one variable version of Taylor's formula to functions in $C^{l,\beta}_{(0)}(\overline{M};\Sigma^2\overline{M})$, with $m \le l$, gives that

$$u(\theta,\rho) = \sum_{i=0}^{m-1} \frac{1}{i!} \cdot \frac{\partial^i}{\partial \rho^i}(\theta,0) + \frac{\rho^m}{(m-1)!}\int_0^1 (1-t)^{m-1} \cdot \frac{\partial^m}{\partial \rho^m}(\theta,t\rho)\,dt.$$

If $u = O(\rho^s)$, all partial derivatives of order less than s should vanish, hence the weighted Hölder space is given by (i). (ii) follows by multiplying through by ρ^{-j}. □

4 Laplace operator

In this section, let's discuss the case when the Laplacian operator is an isomorphism between weighted Hölder spaces.

Proposition 4.1. *Suppose g is asymptotically hyperbolic and consider $f \in C^2(\mathbb{R}), \bar{u} \in C^2(\overline{M})$ and $\kappa \in \mathbb{R}$. We can expand the Laplacian operator apply to $f(\rho)\bar{u}$ as the* *following*

$$(\Delta_g + \kappa)(f(\rho)\,\bar{u}) = (-\rho^2 f''(\rho) + (n-1)\rho f'(\rho) + \kappa f(\rho))\,\bar{u} + \rho X(f),$$

where X is some second-order polynomial in $\rho\frac{d}{d\rho}$.

Let me refer the proof to [1] Corollary 2.7 and 2.8.

Denote

$$I(f) = -\rho^2 f'' + (n-1)\rho f' + \kappa f.$$

The ordinary differential operator is called the *indicial operator* for $\Delta_g + \kappa$ acting on functions. The real numbers s for which $I(\rho^s) = 0$ are called the *characteristic exponents.* For the Laplacian operator, the characteristic exponents are

$$s_{1,2} = \frac{1}{2}(n \pm \sqrt{n^2 + 4\kappa}).$$

It follow from the expansion that

Lemma 4.2. *If $s \neq s_{1,2}$, then there exists a solution $\bar{u} \in C^2(\overline{M})$ to the equation*

$$(\Delta_g + \kappa)(\rho^s \bar{u}) = \rho^s \bar{v} + O(\rho^{s+1}).$$

Proof. Note that $I(\rho^s) = (\kappa - s(s-n))\rho^s$. Taking $\bar{u} = (\kappa - s(s-n))^{-1}\bar{v}$,

$$(\Delta_g + \kappa)(\rho^s \bar{u}) = I_i(\rho^s)\bar{u} + \rho X(\rho^s) = (\kappa - s(s-n))\rho^s(\kappa - s(s-n))^{-1}\bar{v} + O(\rho^{s+1}).$$

□

More generally, there is a theorem for a self-adjoint, elliptic, geometric partial differential operator of order $m \leq l$. For our purpose, it is enough to consider indicial theory for the Laplacian operator. I state the theorem here and refer to [1] for a complete discussion on indicial operators.

Theorem 4.3. *Let $P : C^\infty(M; E) \to C^\infty(M; E)$ be a self-adjoint, elliptic, geometric partial differential operator of order $m \le l$. Suppose $\alpha \in (0,1)$, $k+\alpha \in (0, l+\beta)$, $\delta \in \mathbb{R}$ and $u \in C^{k,\alpha}_{(s)}(\overline{M}; E)$ where $s \in [1, k+\alpha)$. Then*

$$\rho^{-\delta-s} P(\rho^\delta u)\Big|_{\partial M} = I_{\delta+s}(P)\hat{u}.$$

5 Approximation

5.1 First approximate solution

We now have all the tools needed to construct approximate solutions. In this section we will prove that if t is an asymptotically hyperbolic metric then $Q(g,t) = O(\rho^{-1})$. So that this gives our first attempt of the approximation.

Proposition 5.1. *Suppose t is an asymptotically hyperbolic metric and g is conformally compact with $\bar{g} \in C^2(\overline{M}, S^2)$. Then $Q(g,t) = O(\rho^{-1})$ if and only if the following the holds on ∂M*

$$tr_{\bar{g}}(\bar{t}) = n+1, \quad \bar{g}^{-1} d\rho = \bar{t}^{-1} d\rho.$$

Proof. Note that $Q(g,t)$ is of the form

$$\rho^{-2}\mathcal{E}^0(\bar{g},\bar{t}) + \rho^{-1}\mathcal{E}^1(\bar{g},\bar{t}) + \mathcal{E}^2(\bar{g},\bar{t}).$$

So $Q(g,t) = O(\rho^{-1})$ precisely when $\mathcal{E}^0(\bar{g},\bar{t})$ vanishes. Expanding $\mathcal{E}^0(\bar{g},\bar{t})$ gives

$$\mathcal{E}^0(\bar{g},\bar{t}) = n(1 - \bar{g}^{il}\rho_i\rho_j)\bar{g}_{jk} - \frac{1}{2}(B_k\rho_j + B_j\rho_k),$$

where $B = [(tr_{\bar{g}}\bar{t})\bar{g}\bar{t}^{-1} - (n+1)d\rho]$. So the "only if" part is clear (and this is what we need for the first approximation).

For the "if" part, if we set the right hand side to be zero, then it forces $B = 0$. □

Now for a given boundary condition $\hat{g}$ on ∂M, let's denote $g_0 = \rho^{-2}\bar{g}$, and require g_0 to be a metric satisfying the boundary condition. If we fix the auxiliary metric which satisfies the equations in Proposition 5.1, then $Q(g_0, t) = O(\rho^{-1})$, hence giving a first approximate solution g_0 to the equation $Q(g_0, t) = 0$. Here we have an obvious choice $t = g_0$. Through out the rest of our approximation, we will set $t = g_0$.

Remark 5.2. Furthermore, we can use Proposition 5.1 to ensure all approximate solutions in our construction are asymptotically hyperbolic, and satisfy the boundary condition.

5.2 Higher-order approximations

In this section, we will modify the first approximation g_0 inductively to make $Q(g, g_0)$ vanish to a higher order. We require the construction depends smoothly on the initial data since we need to apply the inverse function theorem. The goal is to prove the following theorem.

Theorem 5.3. *Suppose $0 < \beta < 1$, $2 \leq l \leq n-1$, and h is an asymptotically hyperbolic metric on M of class $C^{l,\beta}$. Let $\hat{g}$ be any metric on ∂M of class $C^{l,\beta}$, and set*

$$g_0 = T(\hat{g}) = h + \rho^{-2} E(\hat{g} - \hat{h}).$$ [1]

There exists an asymptotically hyperbolic metric g of class $C^{l,\beta}$ on M such that $\rho^2 g|_{\partial M} = \hat{g}$ and

$$Q(g, g_0) \in C^{l-2,\beta}_{(l-2+\beta)}(\overline{M}; \Sigma^2 \overline{M}).$$

In order to match the notation in [2], let's denote

$$\overline{Q}(\bar{g}, \bar{t}) = \rho^2 Q(\rho^{-2} g, \rho^{-2} g_0).$$

Our plan is to approximate the solution g by a sequence of metrics g_k, such that

$$\overline{Q}(\bar{g}_k, \bar{g}_0) = o(\rho^k).$$

[1]Here we need the help of an extension operator E which extends the metric to the interior and keep the resulting metric asymptotically hyperbolic.

5.2.1 Identify the Hölder space containing Q

Lets first assume such a sequence $\{\bar{g}_k\}$ of approximate solutions exists and see which Hölder space is $\overline{Q}$ belongs to:

The case $k=0$. Note that $\overline{Q}$ has two more orders of ρ. Using the computation for the first approximation, $\overline{Q}(\bar{g}_0,\bar{g}_0)=O(\rho)$. On the other hand, we can write $\overline{Q}$ in local coordinates as the following:

$$\overline{Q}(\bar{g},\bar{g}_0)=\mathcal{E}^0(\bar{g},\bar{g}_0)+\rho\mathcal{E}^1(\bar{g},\bar{g}_0)+\rho^2\mathcal{E}^2(\bar{g},\bar{g}_0).$$

Recall $\mathcal{E}^0(\bar{g},\bar{g}_0)$ are polynomials in $\bar{g}$ and its inverse. If $\bar{g}$ is of class $C^{l,\beta}_{(0)}(\overline{M};\Sigma^2\overline{M})$, $\mathcal{E}^0(\bar{g},\bar{g}_0)\in C^{l,\beta}_{(0)}(\overline{M};\Sigma^2\overline{M})$. Hence

$$\overline{Q}(\bar{g},\bar{g}_0)\in C^{l,\beta}_{(0)}(\overline{M};\Sigma^2\overline{M})+\rho C^{l-1,\beta}_{(0)}(\overline{M};\Sigma^2\overline{M})+\rho^2 C^{l-2,\beta}_{(0)}(\overline{M};\Sigma^2\overline{M}).$$

Intersecting this with $O(\rho)$, and using Hölder embedding (Lemma 3.2 (ii)), the first term (blue) is absorbed by the second term (red). So

$$\begin{aligned}\overline{Q}(\bar{g},\bar{g}_0)\in{}& C^{l,\beta}_{(1)}(\overline{M};\Sigma^2\overline{M})+\rho C^{l-1,\beta}_{(0)}(\overline{M};\Sigma^2\overline{M})+\rho^2 C^{l-2,\beta}_{(0)}(\overline{M};\Sigma^2\overline{M})\\ &\subset\rho C^{l-1,\beta}_{(0)}(\overline{M};\Sigma^2\overline{M})+\rho^2 C^{l-2,\beta}_{(0)}(\overline{M};\Sigma^2\overline{M}).\end{aligned}$$

The case $k=1$. Similar to $k=0$, under the assumption $\overline{Q}(\bar{g}_1,\bar{g}_0)=o(\rho)$, we apply Lemma 3.2 (ii) again to obtain the embedding $\rho C^{l-1,\beta}_{(1)}(\overline{M};\Sigma^2\overline{M})\subset\rho^2 C^{l-2,\beta}_{(0)}(\overline{M};\Sigma^2\overline{M})$. This implies

$$\overline{Q}(\bar{g}_1,\bar{g}_0)\in\rho^2 C^{l-2,\beta}_{(0)}(\overline{M};\Sigma^2\overline{M}).$$

The case $k>1$. In this case we are left with a single Hölder space, so any extra powers of ρ is combined into the weighted part.

To summarize the above: The assumption $\overline{Q}(\bar{g}_k,\bar{g}_0)=o(\rho^k)$ gives the following

$$\overline{Q}(\bar{g}_k,\bar{g}_0)\in\begin{cases}\rho C^{l-1,\beta}_{(0)}(\overline{M};\Sigma^2\overline{M})+\rho^2 C^{l-2,\beta}_{(0)}(\overline{M};\Sigma^2\overline{M}), & k=0\\ \rho^2 C^{l-2,\beta}_{(k-1)}(\overline{M};\Sigma^2\overline{M}), & 1\le k\le l-1\\ \rho^2 C^{l-2,\beta}_{(l-2+\beta)}(\overline{M};\Sigma^2\overline{M}), & k=l\end{cases}$$

5.2.2 Existence of the sequence

It remains to prove such a sequence $\{\bar{g}_k\}$ exists. Assume by induction for some k, we have constructed $\bar{g}_{k-1} \in C^{l,\beta}_{(0)}(\overline{M}; \Sigma^2\overline{M})$ satisfying the above. It suffices to find $\bar{r} \in C^{l,\beta}_{(k)}(\overline{M}; \Sigma^2\overline{M})$, such that $\bar{g}_k = \bar{g}_{k-1} + \bar{r}$.

Using Taylor expansion,

$$\overline{Q}(\bar{g}_{k-1} + \bar{r}, \bar{g}_0) = \overline{Q}(\bar{g}_{k-1}, \bar{g}_0) + D_1\overline{Q}_{(\bar{g},\bar{t})}(\bar{r}) + \int_0^1 (1-\lambda) D_1\overline{Q}_{(\bar{g}+\lambda\bar{r},\bar{t})}(\bar{r}, \bar{r}) d\lambda.$$

Here the second order derivative of $D_1 Q$ is ρ^{-2} times a homogeneous quadratic polynomial in $\bar{r}, \rho\partial\bar{r}$ and $\rho^2\partial^2\bar{r}$, hence of order $o(\rho^k)$. This implies

$$\begin{aligned}\overline{Q}(\bar{g}_k, \bar{g}_0) &= \overline{Q}(\bar{g}_{k-1} + \bar{r}, \bar{g}_0) \\ &= \overline{Q}(\bar{g}_{k-1}, \bar{g}_0) + \rho^2(\Delta_L + 2n)(\rho^{-2}\bar{r}) + o(\rho^k).\end{aligned}$$

Let's denote $\hat{v} = \rho^{-k}\overline{Q}(\bar{g}_{k-1}, \bar{g}_0)\Big|_{\partial M}$. It suffices to find $\bar{r}$ such that

$$\rho^{2-k}(\Delta_L + 2n)(\rho^{-2}\bar{r})\Big|_{\partial M} = I_{k-2}(\Delta_L + 2n)(\rho^{-k}\bar{r})\Big|_{\partial M} + O(\rho).$$

Applying Lemma 4.2, we know when $-2 < s < n-2$, $I_s(\Delta_L + 2n)$ is invertible. So there is a unique $C^{l-k,\beta}$ tensor field ψ along ∂M that solves

$$I_{k-2}(\Delta_L + 2n)(\psi) = -\hat{v}.$$

Then there is a tensor field $\bar{r} \in C^{l,\beta}_{(k)}(\overline{M}; \Sigma^2\overline{M})$ such that

$$\rho^{-k}\bar{r}\Big|_{\partial M} = \psi \implies I_{k-2}(\Delta_L + 2n)(\rho^{-k}\bar{r}) = \rho^2(\Delta_L + 2n)(\rho^{-2}\bar{r}) = -\hat{v}.$$

After the $k = l$ step, we obtain a metric $\bar{g} = \bar{g}_l$ which is lies in the Hölder space $C^{l-2,\beta}_{(l-2+\beta)}(\overline{M}; \Sigma^2\overline{M})$. This completes the proof.

Note that the above proof gives an operator $S : \hat{g} \mapsto g$.

6 Inverse function theorem

In this section, we will apply the inverse function theorem. First, let's restrict the space so that $S(\hat{g}) + r$ is a metric and consider the map

$$\mathcal{Q}(\hat{g}, r) = \Big(\hat{g}, Q(S(\hat{g}) + r, T(\hat{g}))\Big),$$

where $T(\hat{g}) = h + \rho^{-2} E(\hat{g} - \hat{h})$ will be an asymptotically hyperbolic metric, when $\hat{g}$ is closed enough to $\hat{h}$.

Note that when h is Einstein, $S(\hat{h}) = T(\hat{h}) = h$ and hence $\mathcal{Q}(\hat{h}, 0) = (\hat{h}, 0)$. Moreover the linearzation of $\mathcal{Q}$ at $(\hat{h}, 0)$ is given by

$$\begin{aligned} D\mathcal{Q}_{(\hat{h},0)}(\hat{q}, r) &= \Big(\hat{q}, D_1 Q_{(h,h)}(DS_{\hat{h}}\hat{q} + r) + D_2 Q_{(h,h)} DT_{\hat{h}}\hat{q}\Big) \\ &= (\hat{q}, (\Delta_L + 2n)r + K\hat{q}), \end{aligned}$$

where $K\hat{q} = D_1 Q_{(h,h)}(DS_{\hat{h}}\hat{q}) + D_2 Q_{(h,h)} DT_{\hat{h}}\hat{q}$.

Use Lemma 4.2 we know that $\Delta_L + 2n$ is invertible, hence the linearization of $\mathcal{Q}$ is nonsingular. We can actually write its inverse explicitly

$$(D\mathcal{Q}_{(\hat{h},0)})^{-1}(\hat{w}, v) = (\hat{w}, (\Delta_L + 2n)^{-1})(v - K\hat{w}).$$

So we are allowed to apply the inverse function theorem, and get a solution to the equation $Q(g, t) = 0$.

Now it remains to use algebraic argument to prove that when we have a solution of $Q(g, t) = 0$ and the Ricci curvature strictly negative on the manifold. That is, when

$$\mathrm{Ric}_g(V, V) \leq K|V|_g^2, \text{ for some } K \leq 0,$$

the perturbed operator Φ vanishes, so that we obtain an Einstein metric.

Apply Bianchi operator to the equation $Q(g, t) = 0$ gives $\delta_g G_g \Phi(g, t) = 0$. If we write $w = gt^{-1}\delta_g G_g t$ then this becomes $\delta_g G_g \delta^* w = 0$. Using Ricci identity this can be writen as $\frac{1}{2}(w^j_{i,j} + R_{ij}w^j) = 0$. Hence we can bound the Laplacian by

$$\Delta_L |w|_g^2 \leq K|w|_g^2.$$

And using generalized maximum principle we conclude $w = 0$.

References

[1] Robin Graham and John M Lee, *Einstein Metrics with Presribed Conformal Infinity on the Ball*, 1991.

[2] John M Lee, *Fredholm Operators and Einstein Metrics on Conformally Compact Manifolds*, American Mathematical Society, 2006.

[3] Dennis M. DeTurck, *Existence of Metrics with Prescribed Ricci Curvature: Local Theory*, Inventiones mathematicae, 1981.