The Renormalized Volume of Conformally Compact Einstein Manifolds

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Oct 12, 2022

Abstract

In this talk, I will introduce the renormalized volume of a conformally compact Einstein manifolds. The classical volume for any conformally compact manifold is infinite, just like the case for a hyperbolic plane. We are interested in finding an appropriate renormalization. It turns out that under Einstein condition, the zeroth order term in the volume expansion of the complement of a collar neighborhood gives a scalar conformal invariant. In the even-dimensional case, this term is the renormalized volume.

This renormalization is initially motivated by the AdS/CFT correspondence in physics. There are many interesting results of the renormalized volume of a conformally compact manifold. For example, we can link the renormalization to the Chern-Gauss-Bonnet formula and Branson's Q-curvature. Furthermore, we may define a renormalized integral and prove a renormalized version of the Atiyah-Singer index theorem.

Contents

1 Introduction

In the first two sections we follow the construction in Graham's paper [5] to define the renormalized volume . After that, we will see examples of linking the renomalization with Gauss-Bonnet theorem.

1.1 Motivation

- Volume of conformally compact manifold is unbounded. Certain renormalization is required to obtain a geometric invariants of conformally compact manifold.
- In physics, one associate observableS to submanifolds N in M. Using a suitable approximation, AdS/CFT correspondence in physics offers a way to compute the expectation of an observable in terms of the volume of minimal submanifolds Y whose boundary is N.
- The coefficient before log term (n odd case) gives a generalized version of the Willmore functional ("the rigid sting action") on conformal manifold.
- There is a renormalize version of the Atiyah-Singer index theorem.

1.2 Set up

Through out this notes, we let $\bar{X}^{n+1}$ be a manifold with boundary, and denote X as its interior, and M as its boundary.

Definition 1.1 (bdf)**.** A *boundary defining function* (bdf) is a smooth function ρ on $\bar{X}$, which is positive on X and vanishes to the first order on M.

Definition 1.2 (conformally compact)**.** A Riemannian metric g_+ on X is called *conformally compact* if for some choice of bdf ρ, $\bar{g} := \rho^2 g_+$ extends continuously as a metric to $\bar{X}$.

Definition 1.3 (conformal infinity)**.** Let g_+ be Riemannian metric on X, and let h be Riemannian metric on M. The conformal class $[h]$ is called the *conformal infinity* of g_+, if for some choice of bdf ρ, $\bar{g} := \rho^2 g_+$ extends continuously as a metric to $\bar{X}$ and $\bar{g}|_M = h$.

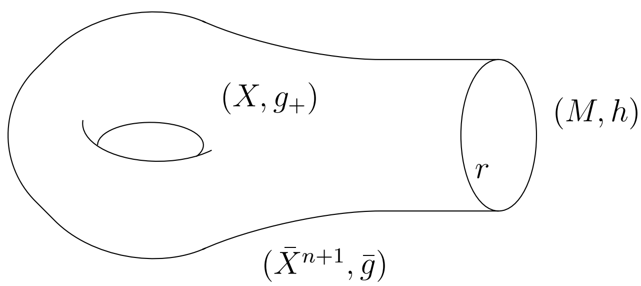

Figure 1: Manifold with boundary, bdf and conformal infinity

Example 1.4.

1. Hyperbolic plane. Consider $\mathbb{H}$ with the hyperbolic metric $g_+ = \dfrac{dx^2 + dy^2}{y^2}$. Here the bdf is y, with conformal infinity $h = dx^2$.
2. Hyperbolic ball. Consider B^{n+1} with the hyperbolic metric

$$g_+ = g_{B^{n+1}} = \frac{4\sum_i (dx^i)^2}{(1-|x|^2)^2}.$$

   Here the bdf is $\dfrac{(1-|x|^2)^2}{2}$, with conformal infinity $h = \sum_i (dx^i)^2|_{S^n}$.

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From now on we assume g_+ is Einstein, i.e. $\mathrm{Ric}^{g_+} = -ng_+$. This condition determines the bdf uniquely. Indeed, under conformal change we may write

$$\mathrm{Ric}_{ij} = -|\mathrm{d}\rho|^2_{\bar{g}}\, n\, g_{ij} + O(\rho^{-3}),$$

where $|\mathrm{d}\rho|^2_{\bar{g}} = \bar{g}^{ij} r_i r_j$. So the Einstein condition implies $|\mathrm{d}\rho|^2_{\bar{g}} = 1$. Then it follows from the fact that for $\rho = e^w x$, the PDE

$$1 = |\mathrm{d}\rho|^2_{\bar{g}} = |dx + xdw|^2_{\bar{g}} + 2x(\nabla_{\bar{g}} x)w + x^2 |dw|^2_{\bar{g}}$$

has unique solution.

Definition 1.5. We call the conformally compact metric g on M *asymptotically hyperbolic* if the bdf ρ satisfies $|\mathrm{d}\rho|^2_{\bar{g}} = 1$. And ρ is called a *special bdf*.

Consider a collar neighborhood $M \times [0, \epsilon)$ of M, where the metric $\bar{g}$ takes the normal form $g_\rho + \mathrm{d}\rho^2$. Hence

$$g_+ = \rho^{-2}(g_\rho + \mathrm{d}\rho^2). \tag{1}$$

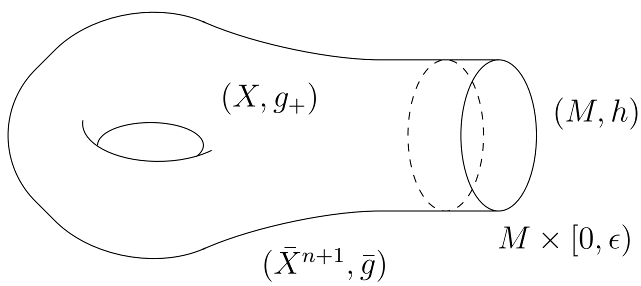

Figure 2: Collar neighborhood

Example 1.6 (Special bdf for hyperbolic ball)**.** The special bdf for the hyperbolic metric $g_{B^{n+1}}$ is $\rho = \dfrac{1-|x|}{1+|x|}$, and $\bar{g} = \dfrac{4\sum_i (dx^i)^2}{(1+|x|)^4}$ can be decomposed as

$$\bar{g} = \underbrace{\frac{(1-\rho^2)^2}{4} g_{S^n}}_{g_\rho} + \mathrm{d}\rho^2.$$ •

2 Volume and area renormalization

2.1 Volume renormalization

In this section we defined the renormalized volume.

Using Equation (1) the volume form dvol_{g_+} is given by

$$\mathrm{dvol}_{g_+} = \rho^{-n-1}\sqrt{\frac{\det g_\rho}{\det h}}\,\mathrm{dvol}_h \mathrm{d}\rho. \tag{2}$$

Substitute into the volume integral below we have

$$\mathrm{Vol}_{g_+}(\{\rho > \epsilon\}) = \int_{\{\rho>\epsilon\}} \mathrm{dvol}_{g_+} = \int_\epsilon^\infty \rho^{-n-1}\int_M \sqrt{\frac{\det g_\rho}{\det h}}\,\mathrm{dvol}_h \mathrm{d}\rho. \tag{3}$$

Example 2.1 (4D hyperbolic ball [8])**.** Let $(X^{n+1}, g_+) = (B^4, g_{B^4})$. Recall from Example 1.6, we have

$$\rho = \frac{1-|x|}{1+|x|},\ \ h = \frac{1}{4}g_{S^3} \ \text{ and } \ g_\rho = \frac{(1-\rho^2)^2}{4}g_{S^3}.$$

Substitute into Equation (3) yields,

$$\begin{aligned}
\mathrm{Vol}_{g_+}(\{\rho > \epsilon\}) &= \int_{\{\rho>\epsilon\}} \mathrm{dvol}_{g_+} \\
&= \int_\epsilon^1 \rho^{-4}\int_{S^3}\sqrt{\frac{\det g_\rho}{\det h}}\,\mathrm{dvol}_h \mathrm{d}\rho \\
&= \int_\epsilon^1 \rho^{-4}\int_{S^3}(1-\rho^2)^3\sqrt{\frac{\det g_{S^3}}{\det g_{S^3}}}\,\frac{1}{8}\,\mathrm{dvol}_{g_{S^3}}\mathrm{d}\rho \\
&= \frac{\mathrm{Area}(S^3)}{8}\int_\epsilon^1 \rho^{-4}(1-\rho^2)^3\mathrm{d}\rho \\
&= \frac{\mathrm{Area}(S^3)}{8}\left(\frac{(1-\epsilon^2)^3}{3\epsilon^3} - \frac{2(1-\epsilon^2)^2}{\epsilon} + \frac{8}{3} - 4\epsilon - \frac{4\epsilon^3}{3}\right).
\end{aligned}$$

Note that the constant term is $\dfrac{\mathrm{Area}(S^3)}{3}$, which does not depend on the choice of special bdf's. •

We now decompose the volume above using the following the Fefferman-Graham expansion of g_ρ under Einstein condition (for detail see [5]):

$$g_\rho = \begin{cases} g_0 + g_2\rho^2 + (\text{even powers}) + g_{n-1}\rho^{n-1} + g_n\rho^n + \cdots & n \text{ odd} \\ g_0 + g_2\rho^2 + (\text{even powers}) + g_{n,1}\log(\rho)\rho^{n-1} + g_n\rho^n + \cdots & n \text{ even.} \end{cases}$$

Taking $g_0 = g$, we may write the square root part as

$$\sqrt{\frac{\det g_\rho}{\det g}} = 1 + v_2\rho^2 + (\text{even powers}) + v_n\rho^n + o(\rho^n), \tag{4}$$

where v_j are locally determined functions on M and $v_n = 0$ for n odd. Then the asymptotic expansion of $\mathrm{Vol}_{g_+}(\{\rho > \epsilon\})$ as $\epsilon \to 0$ is

$$\begin{aligned} &\mathrm{Vol}_{g_+}(\{\rho > \epsilon\}) \\ &= \begin{cases} c_0\epsilon^{-n} + c_2\epsilon^{-n+2} + \ (\text{odd powers}) \ + c_{n-1}\epsilon^{-1} + V + o(1) & n \text{ odd} \\ c_0\epsilon^{-n} + c_2\epsilon^{-n+2} + \ (\text{even powers}) \ + c_{n-2}\epsilon^{-2} + L\log\frac{1}{\epsilon} + V + o(1) & n \text{ even.} \end{cases} \end{aligned}$$

Here all the coefficients c_{2k} and L are integrals over M of local curvature expressions of g. Explicitly,

$$c_{2k} = \frac{1}{n-2k}\int_M v_{2k}\,\mathrm{dvol}_g \quad \text{and} \quad L = \int_M v_n\,\mathrm{dvol}_g.$$

Definition 2.2. The *renormalized volume* ${}^R\mathrm{Vol}(g)$ is defined to be the zero-th order term V in the above expansion.

Example 2.3.

1. Take $n = 2$. One can compute $v_2 = -\dfrac{R}{4}$ and by Gauss-Bonnet theorem we have

   $$L = \int_M v_2\,\mathrm{dvol}_g = -\pi\chi(M).$$

   The shows that L is an invariant, whereas ${}^R\mathrm{Vol}$ is not:

   $${}^R\mathrm{Vol}(g) - {}^R\mathrm{Vol}(e^{2w}g) = \int -\frac{Rw + w_i w^i}{4}\,\mathrm{dvol}_g.$$

2. For $n = 3$ and g also asymptotically hyperbolic [4]. One can compute
$$6\,{}^R\mathrm{Vol}_{g_+} = 8\pi^2\chi(M) - \frac{1}{4}\int_M |W|^2\,\mathrm{dvol}_h.$$

3. For $n = 4$, we have
$$L = \int_M v_4\,\mathrm{dvol}_g = \int_M \frac{(P_i^i)^2 - P_{ij}P^{ij}}{8}\,\mathrm{dvol}_g = \frac{\pi^2\chi(M)}{2} - \int_M \frac{1}{64}|W|^2\,\mathrm{dvol}_h,$$
where W and P denote the Weyl and Schouten tensor respectively.

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Remark 2.4.

- As it suggested in the above examples, for n even, the zero-th order term V depends on the choice of g (equivalently, depends on the choice of special bdf ρ), whereas the log term coefficient L does not.
- This dependence on ρ is mediated through $g_{n,1}$ in the Fefferman-Graham expansion.

Theorem 2.5. *If n is odd, then V is a conformal invariant. If n is even, then L is a conformal invariant.*

Proof. (For detail see [5], Theorem 3.1). For odd n, take two special bdf ρ and $\hat{\rho}$, with corresponding metric g and $\hat{g}$. Consider the difference
$$\mathrm{Vol}(\{\rho > \epsilon\}) - \mathrm{Vol}(\{\hat{\rho} > \epsilon\}).$$

Step 1. Convert this difference into an integral over M cross an interval.

Recall Equation (1.2) which tells the relation between these two bdf's. One can solve ρ in terms of $\hat{\rho}$, and hence
$$\mathrm{Vol}(\{\rho > \epsilon\}) - \mathrm{Vol}(\{\hat{\rho} > \epsilon\}) = \int_M \int_{(\epsilon,\hat{\epsilon})} \mathrm{dvol}_{g_+}.$$

Step 2. Now evaluate the above integral using Fefferman-Graham expansion.

Check that
$$\int_M \int_{(\epsilon,\hat{\epsilon})} \mathrm{dvol}_{g_+} = \sum_{0\le j\le n, j \text{ even}} \int_M \frac{v_j(x)}{-n+j}\,(\text{even terms}^1)\,\mathrm{dvol}_g + o(1).$$
Now let's compare the zero-th order term when $\epsilon \to 0$, Left hand side gives the difference ${}^R\mathrm{Vol}(g) - {}^R\mathrm{Vol}(\hat{g})$, whereas right hand side does not have any constant term. □

2.2 Area renormalizaton

The renormalized area is defined using a similar idea. Let's briefly discuss it.

Consider a minimal surface $Y \subset X$ of dimension $k+1$. Set the boundary of Y to be $N = \bar{Y} \cap M$, which is a submanifold of M. Locally near a point in N, we take (x, u) to be the coordinate on M, with $N = \{u = 0\}$. Let ρ be a bdf of M.

Now we may write Y as the graph $\{u = u(x, \rho)\}$. The asymptotics of $u(x, \rho)$ as $r \to 0$ is quite similar to the expansion we have for g_ρ:

$$u = \begin{cases} u_2\rho^2 + (\text{even powers}) + u_{k+1}\rho^{k+1} + u_{k+2}\rho^{k+2} + \cdots & n \text{ odd} \\ u_2\rho^2 + (\text{even powers}) + u_k\rho^k + u_{k,1}\log(\rho)\rho^{k+2} + u_{k+2}\rho^{k+2} + \cdots & n \text{ even} \end{cases}$$

where u_j are locally determined as functions of x, except for u_{k+2}.

Similarly we have expansion of area from as

$$\mathrm{dA}_Y = \rho^{-k-1}\Big(1 + A_2\rho^2 + (\text{even powers}) + A_k\rho^k + o(\rho^k)\Big)\,\mathrm{dA}_N\mathrm{d}\rho,$$

where a_j are locally determined functions on N and $a_k = 0$ for k odd.

The asymptotic expansion of $\mathrm{Vol}_{g_+}(\{\rho > \epsilon\})$ as $\epsilon \to 0$ is

$$\begin{aligned} &\mathrm{Area}(Y \cap \{\rho > \epsilon\}) \\ &= \begin{cases} b_0\epsilon^{-k} + b_2\epsilon^{-k+2} + \text{ (even powers) } + b_{k-1}\epsilon^{-1} + A + o(1) & n \text{ odd} \\ b_0\epsilon^{-k} + b_2\epsilon^{-k+2} + \text{ (even powers) } + b_{k-2}\epsilon^{-2} + K\log\frac{1}{\epsilon} + A + o(1) & n \text{ even} \end{cases} \end{aligned}$$

Here all the coefficients b_i and K are integrals over N of local curvature expressions of g. In particular, $K = \displaystyle\int_N a_n \mathrm{dA}_N$.

3 Integral renormalization

In this section we introduce another regularization and compare it with the renormalization we have from above. We will follow the discussion in [2].

The renormalization we used above is known as Hadamard regularization. This is used in the renormalize version of the Atiyah-Singer index theorem. In order to distinguish with another regularization, we denote it as

$$^H\!\!\int \mu = \operatorname*{FP}_{\epsilon=0} \int_{\rho>\epsilon} \mu,$$

where μ stands for phg density (defined below).

Definition 3.1 (polyhomogenous)**.** We call functions with an expansion of the form

$$\sum_{k \geq k_0} \sum_{p=0}^{p_k} a_{k,p} x^k \log^p x$$

with $a_{k,p}$ smooth independent of x *polyhomogenous* (phg).

We will assume all the densities are phg.

3.1 Riesz regularization

Another approach we may take is the Riesz regularization. Given a bdf, we meromorphically extending the $\zeta_\rho(z) = \int \rho^z \mu$ and define the Riesz renomalization by the finite part at $z = 0$,

$$^R\!\!\int \mu = \operatorname*{FP}_{z=0} \zeta_\rho(z).$$

For concreteness, consider the case $\mu = \mathrm{dvol}_{g_+}$.

Take $\zeta_\rho(z) = \int_X \rho^z \,\mathrm{dvol}_g$. Note that this integral converges if and only if $\mathrm{Re}(z) > n-1$. So $\zeta_\rho(z)$ is holomorphic on a half plane, and it has a meromorphic continuation to $\mathbb{C}$.

Consider $\mathrm{dvol}_g = f(\rho, y)\,\mathrm{dvol}_{g_+}\mathrm{d}\rho$ for some $f(\rho, y)$ which has a Taylor expansion. Let $a_j(y)$ be the coefficients in Taylor expansion so that we can break $\zeta_\rho(z)$ into three parts:

$$\begin{aligned} \zeta_\rho(z) = \int_{\{\rho > \epsilon\}} \rho^z \,\mathrm{dvol}_{g_+} + \int_{M\times[0,\epsilon)} \left(f(\rho, y) - \sum_{j=0}^{N} a_j(y)\rho^j \right) \rho^{-n}\,\mathrm{dvol}_{g_0}\mathrm{d}\rho \\ + \int_{M\times[0,\epsilon)} \left(\sum_{j=0}^{N} a_j(y)\rho^j \right) \rho^{-n}\,\mathrm{dvol}_{g_0}\mathrm{d}\rho \qquad =: I + II + III. \end{aligned}$$

Here g_0 is the metric appear in the expansion of g_ρ.

One may check III has the form

$$\sum_{j=0}^{N} \frac{A_j}{z + j - n + 1} \epsilon^{z+j-n+1},$$

and $II = O(\rho^{N+1})$ is holomorphic if $\mathrm{Re}(z + N + 1 - n) > -1$. Hence, a meromorphic continuation of $\zeta_\rho(z)$ exists, and the poles are at $-j + n - 1, j \in Z_{\geq 0}$. So it make sense to take the zero-th order term as the finite part and we define the renormalized volume using Riesz regularization as

$$ {}^{R}\mathrm{Vol} = \mathop{\mathrm{FP}}_{z=0} \zeta_\rho(z). $$

As a note, this construction can be generalize to any phg densities, which gives the so-called renormalized integral.

We now compare the Hadmard and Riesz renormalizations on phg densities. For $k \neq -1$, we have

$$ {}^{H}\!\!\int_{[0,\epsilon)} \rho^k \log^p(\rho)\,\mathrm{d}\rho = {}^{R}\!\!\int_{[0,\epsilon)} \rho^k \log^p(\rho)\,\mathrm{d}\rho = \epsilon^{k+1}\sum_{l=0}^{p} c_l \log^{p-l}(\rho)\epsilon. $$

For $k = -1$, these two integrals give different answers:

$$ {}^{H}\!\!\int_{[0,\epsilon)} \frac{\log^p \rho}{\rho}\,\mathrm{d}\rho = \frac{\log^p \rho}{p+1}\epsilon \quad \text{whereas} \quad {}^{R}\!\!\int_{[0,\epsilon)} \frac{\log^p \rho}{\rho}\,\mathrm{d}\rho = 0. $$

Hence

$$ {}^{R}\mathrm{Vol}(X) = \mathop{\mathrm{FP}}_{z=0} \int_X \rho^z \,\mathrm{dvol}_{g_+} = \mathop{\mathrm{FP}}_{\epsilon=0} \int_{\{\rho>\epsilon\}} \mathrm{dvol}_{g_+}. $$

4 Applications

We have already seen there is a link between the Euler characteristic $\chi(M)$ and the conformal invariant L defined in Section 1. Next let me state several result using the renormalized integral.

4.1 Pfaffian

On an even-dimensional asymptotically hyperbolic manifold $\bar{X}$, with $\bar{g} = \mathrm{d}\rho^2 + g_\rho$ and $tr_{g_0} g_n = 0$ (here g_0 and g_n comes from the expansion of g_ρ), we have

$$ {}^{R}\!\!\int \mathrm{Pff} = \chi(M). $$

This follows from applying the Chern-Gauss-Bonnet theorem for manifold with boundary:

$$ \int_{\{\rho>\epsilon\}} \mathrm{Pff} + \int_{\{\rho=\epsilon\}} II = \chi(\{\rho>\epsilon\}) = \chi(M). $$

The vanishing of the trace implies the second term in Chern-Guass-Bonnet vanishes.

4.2 Renormalized index theorem

Similarly, we may formulate the index theorem using renormalization [1]. The index theorem of a Dirac-type operator $\eth$ on a manifolds with boundary is

$$\int AS - \frac{1}{2}\eta(M) = \mathcal{H} + \operatorname{ind}(\eth),$$

where $\mathcal{H}$ is some extended solution. [2]

Using renormalized integral, the above takes the form [3]

$${}^{R}\!\!\int AS - \frac{1}{2}{}^{R}\eta(M) = \lim_{t\to\infty} {}^{R}\mathrm{Str}(e^{-(t\eth^{E})^2}).$$

If we assume further that $\mathrm{Im}(\eth^2)$ is closed, then the right hand side is ${}^{R}\mathrm{ind}(\eth)$.

Analogous to the classical case, the renormalized Gauss-Bonnet theorem is a special case for the renormalized index theorem.

5 Generalization

5.1 Singular Yamabe metrics

One may generalize this volume renormalizaton process to singular Yamabe metrics [6], where Einstein condition is replaced by finding a defining function ρ of M such that $g_+ = \rho^{-2}\bar{g}$ has constant scalar curvature. Using transition formula for scalar curvature under conformal change. One may transfer the problem into solving a PDF of a form similar to Equation (1.2).

Volume expansion has a similar pattern, and the of log term coefficient, if we call it as L again, is the obstruction for this singular Yamabe problem.

5.2 Other known results

Two other known results are: In dimension 4, there is a well defined renormalized volume if (X, g_+) is asymptotically hyperbolic (that is, $|\rho|^2_{\bar{g}} = 1$ on M) and there is a totally geodesic compactification [7]. There is a Fefferman-Graham expansion for g_ρ if we replace the Einstein condition with Lovelock condition [3, Section 2.3], though results for volume renormalization seem to be unknown.

[2] This is denoted as h in [1]. To distinguish from the boundary metric, we use $\mathcal{H}$.

[3] One need to introduce Edge metrics and half distributions to make this statement precisely. This is beyond the scope of this notes. For detail see [1].

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