Introduction to Einstein-Maxwell Equations

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Abstract

Maxwell's equations reveal the fundamental relations between electricity and magnetism. They can be formulated in various equivalent ways under the idea of unification. In this talk, I will formulate Maxwell's equations using differential forms and tensor calculus. We will see there are at least two benefits of tensorizing spacetime, electric-magnetic field, and energy-momentum: this provides a covariant formulation of Maxwell's equation; and, we obtain a crucial term that appears in Einstein field equation of general relativity.

Contents

1 Introduction

In this section, we formulate Maxwell's equations from a different point of view and introduce tensor calculus and Riemannian metric. More precisely, we will do the following

$$\begin{tikzcd}
\text{Differential geometric formulation of Maxwell's equations on } \mathbb{R}^3 \text{ (vector calculus)} \arrow[d, "\text{unify spacetime; electric and magnetic field}"] \\
\text{Tensor calculus formulation of Maxwell's equations} \arrow[d, "\text{identify } E, B \text{ with forms and using Hodge star}"] \\
\text{Hodge-Maxwell theorem} \arrow[d, "\text{perturbing the metric tenosr}"] \\
\text{Maxwell's equations in curved spacetime}
\end{tikzcd}$$

1.1 A brief history

1784-1786 Coulomb did experiments on electric repulsion and attraction (Coulomb force)

1837 Faraday studied Coulomb's law and discovered electro-magnetic induction (electric field lines)

1855, 1861, 1864 Maxwell formulated Faraday's experiments using mathematical language

1880 Heaviside introduced permittivity ϵ_0

2 Differential geometric formulation

We will start with the differential form formulation of Maxwell's equations in $\mathbb{R}^3$.

$$\text{Gauss' law: } \nabla \cdot E = \frac{\rho}{\epsilon_0} \tag{E}$$

$$\nabla \cdot B = 0 \tag{M}$$

$$\text{Faraday's law of induction: } \nabla \wedge E = -\frac{\partial B}{\partial t} \tag{MI}$$

$$\text{Ampère's circuital law: } \nabla \wedge B = \mu_0 J + \frac{1}{c^2}\frac{\partial E}{\partial t} \tag{EI}$$

Here

- $\rho =$ charge density
- $\epsilon_0 = \frac{1}{4\pi} \cdot \frac{1}{9 \cdot 10^9}\ \mathrm{Fm}^{-1}$, the permittivity of free space
- $J =$ current density
- $\mu_0 = \frac{4\pi}{10^7}\ \mathrm{NA}^{-2}$, the permeability of free space
- $\mu_0 \epsilon_0 = \frac{1}{c^2}$

2.1 Meaning of the equations

- Equation (E): electric charges cause electric fields.
- Equation (M): magnetic monopole does not exist.
- Equation (MI): changing magnetic field induces electric fields.
- Equation (EI): electric current causes magnetic field and changing electric field induces magnetic fields.

Remark 2.1. The second term, which is known as Maxwell's addition, is a correction to resolve compatibility issue with the continuity equation. Let's explore this briefly.

2.2 Maxwell's addition

The continuity equation (also known as electric charge conservation) says

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot J.$$

Hence for any volume V, the total charge $Q = \int_V \rho \, dV$ satisfies

$$\frac{\partial Q}{\partial t} = \int_{\partial V} -\nabla \cdot J \, dA.$$

Note that Equation (M) and Equation (MI) are consistent by taking divergence; whereas Equation (E) and Equation (EI) without Maxwell's addition both hold only if $J = 0$. Maxwell's addition is designed to get rid of this restriction.

Moreover, if we move time derivatives to the left hand side and rearrange the equations into two groups.

homogeneous	inhomogeneous
$\nabla \cdot B = 0$	$\nabla \cdot E = \frac{\rho}{\epsilon_0}$
$\nabla \wedge E + \frac{\partial B}{\partial t} = 0$	$\nabla \wedge B - \frac{1}{c^2}\frac{\partial E}{\partial t} = \mu_0 J$

The reason for rearrangement will become clear when we convert Maxwell's equation into tensor form.

3 Tensor calculus formulation

3.1 Levi-Civita symbol and 4-vectors

Consider the *Levi-Civita symbol*

$$\epsilon_{ijk} = \begin{cases} 1 & (i\ j\ k) \text{ even permutations of } (1\ 2\ 3) \\ -1 & (i\ j\ k) \text{ odd permutations of } (1\ 2\ 3) \\ 0 & \text{otherwise} \end{cases}$$

It has properties

$$(v \times w)_k = \epsilon_{ijk} v^j w^k.$$
$$\epsilon_{ija}\epsilon_{akl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} \implies \epsilon_{ijk}\epsilon_{ijk} = 6.$$

Take the *spacetime coordinate* $x^\mu = (ct, x)$ and define $\partial_\mu = (\frac{1}{c}\partial_t, \nabla)$. (We need to multiply t by speed of light c to match the unit of space). Rotations in $\mathbb{R}^3$ are generalized by Lorentz transformations in spacetime. Let

$$\eta_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then $\eta^{\mu\nu}\partial_\mu\partial_\nu = -\frac{1}{c^2}\partial_t^2 + \Delta$. ($\Delta$ with plus sign).

In this section, we will only consider flat spacetime, which is also known as *Minkowski spacetime*. It seems not necessary to introduce $\eta^{\mu\nu}$, yet the benefit is that we may perturb $\eta^{\mu\nu}$ by a symmetric 2-tensor $h^{\mu\nu}$ and all the above terms work in curved spacetime.

Note that -1 in $\eta_{\mu\nu}$ reflects the crucial difference between ct and x. A coordinate change purely in space corresponds to the usual rotation in $\mathbb{R}^3$, e.g.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

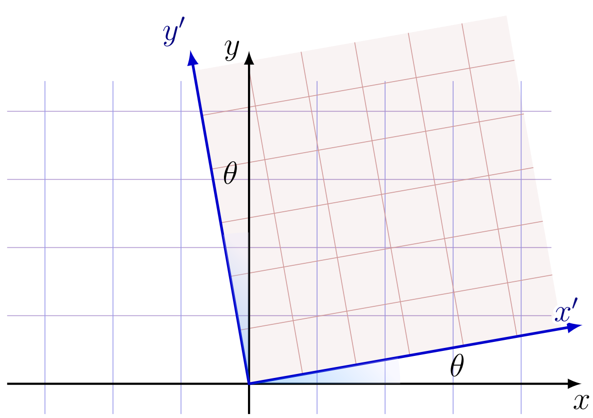

Figure 1: Lorentz transformation on (x, y)-plane

Whereas coordinate change in ct and x corresponds to

$$\begin{bmatrix} \cosh\theta & -\sinh\theta & 0 & 0 \\ -\sinh\theta & \cosh\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

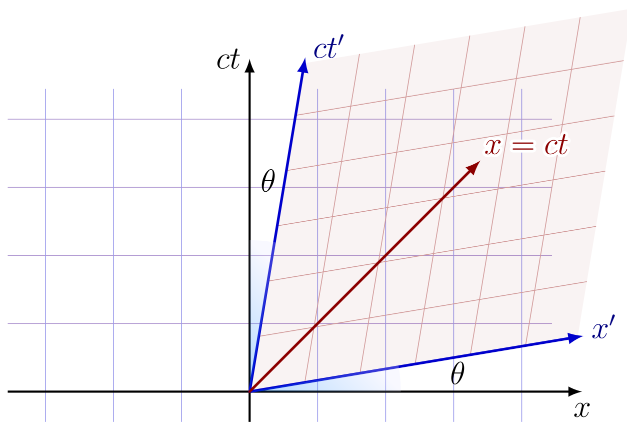

Figure 2: Lorentz transformation on (ct, x)-plane

Definition 3.1. We define the *4-vector current* J^μ and *field strength tensor* $F_{\mu\nu}$[1] to be

$$\begin{aligned} J^\mu &= (\rho c, J), \\ F^{\mu\nu} &= \begin{bmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{bmatrix}. \end{aligned}$$

Note that

$$\begin{aligned} F_{0i} &= -\frac{E_i}{c}, \\ F_{ij} &= \epsilon_{ijk} B_k. \end{aligned}$$

[1] This is an antisymmetric $(0, 2)$-tensor

3.2 Maxwell's equations in tensor form

Using 4-vectors we may formulate the Maxwell's equations as

$$F_{[\mu\nu,\gamma]} = 0 \quad \text{or} \quad \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta,\mu} = 0. \tag{M*}$$

$$F^{\mu\nu}{}_{,\mu} = \mu_0 J^\nu, \tag{E*}$$

If we introduce $\frac{1}{2}G^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$, then we can simplify Equation (M*) further.

homogeneous	inhomogeneous
$G^{\mu\nu}{}_{,\mu} = 0$	$F^{\mu\nu}{}_{,\mu} = \mu_0 J^\nu$

Homogeneity suggests that Equation (M) and (MI) comes from the one on left and Equation (E) and (EI) come from the one on right.

Proposition 3.2. *The second equation with $\mu, \nu, \gamma = i, j, k$ gives Equation (M) and with $\mu, \nu, \gamma = 0, j, k$ gives Equation (MI).*

Proof. Indeed, multiplying the equation by ϵ_{ijk} and using $-2B_i = \epsilon_{ijk}F^{jk}$ we obtain

$$0 = \epsilon_{ijk}F_{[ij,k]} = 3\epsilon_{ijk}F_{ij,k} = 3 \cdot (2\nabla \cdot B).$$

For the second half of the statement, starting with $F_{0j,k} + F_{k0,j} + F_{jk,0} = 0$. Now substitute the values of F_{jk} gives

$$\frac{1}{c}\Big(- E_{j,k} + E_{k,j} + \epsilon_{kjl}\frac{\partial B_l}{\partial t}\Big) = 0.$$

Multiplying the equation by ϵ_{ijk} we obtain

$$0 = \frac{(2\nabla \wedge E)_i}{c} + \frac{2}{c}\frac{\partial B_i}{\partial t}.$$

□

3.3 4-potentials

Definition 3.3. We define the *4-potential* A^μ to be

$$A^\mu = (\frac{\phi}{c}, A).$$

Here

- ϕ = electrostatic potential, defined by $E = -\nabla\phi$.
- A = magnetic vector potential, defined in magnetostatics by $B = \nabla \wedge A$.
- In electrostatics, there is no time dependence, so $\nabla \wedge E = 0$. This means we can write E as gradient of some function which is the potential. Similar for B.

Using 4-potential

$$F^{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

and using Maxwell's equation we obtain

$$B = \nabla \wedge A,$$
$$E = -\nabla\phi - \frac{\partial A}{\partial t}.$$

Note that 4-potential is transformed as

$$\tilde{A}^\mu = A^\mu + \partial^\mu \lambda.$$

We can check

$$\begin{aligned}\tilde{F}^{\mu\nu} &= \partial^\mu \tilde{A}^\nu - \partial^\nu \tilde{A}^\mu = \partial^\mu (A^\nu + \partial^\nu \lambda) - \partial^\nu (A^\mu + \partial^\mu \lambda) \\ &= \partial^\mu A^\nu + \partial^\mu \partial^\nu \lambda - \partial^\nu A^\mu - \partial^\nu \partial^\mu \lambda = \partial^\mu A^\nu - \partial^\nu A^\mu = F^{\mu\nu}\end{aligned}$$

3.4 Energy-momentum conservation

From physics we have

$$\text{Energy density: } \rho_{en} = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0} B^2,$$
$$\text{energy flux: } J_{en} = \frac{1}{\mu_0} E \wedge B.$$

This implies $\frac{\partial \rho_{en}}{\partial t} = -\nabla J_{en}$.

We now introduce the energy-momentum tensor, which plays a key role in the Einstein field equation.

Definition 3.4. We define *energy-momentum tensor* $T^{\mu\nu}$ to be

$$T^{\mu\nu} = \frac{1}{\mu_0} F^{\alpha\mu} F_{\alpha}{}^{\nu} - \frac{1}{4\mu_0} \eta^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}.$$

When we restrict to space, T^{ij} is known as the *Maxwell stress tensor.*

Remark 3.5.

1. Note that $T^{00} = \rho_{en}$, $T^{0i} = J^i_{en}$ and $T^{\mu\nu}{}_{,\mu} = 0$.
2. The Newtonian energy is given by integrating T^{00}: $\int_V T^{00} dx$, whose value is proportional to the first components of the *4-potential* $p^\mu = (\frac{E}{c}, p)$. So entries of $T^{\mu\nu}$ has physical dimensions of energy density.
3. Computing the energy of a stationary time-like object (e.g. a massive particle) gives $E = mc^2$.

Proposition 3.6. *The divergence of the non-gravitational stress–energy $T^{\mu\nu}{}_{,\mu}$ vanishes. This means non-gravitational energy and momentum are conserved.*

Proof.

$$\begin{aligned} T^{\mu\nu}{}_{,\mu} &= \frac{1}{\mu_0} F^{\alpha\mu} F_{\alpha}{}^{\nu}{}_{,\mu} - \frac{1}{2\mu_0} F^{\alpha\beta,\nu} F_{\alpha\beta} \\ &= \frac{1}{2\mu_0} F^{\alpha\mu} (F_{\alpha}{}^{\nu}{}_{,\mu} - F_{\mu}{}^{\nu}{}_{,\alpha}) - \frac{1}{2\mu_0} F^{\alpha\mu,\nu} F_{\alpha\mu} \\ &= \frac{1}{2\mu_0} F^{\alpha\mu} (F_{\alpha}{}^{\nu}{}_{,\mu} + F^{\nu}{}_{\mu,\alpha}) + \frac{1}{2\mu_0} F^{\alpha\mu,\nu} F_{\mu\alpha} \\ &= \frac{1}{2\mu_0} F^{\alpha\mu} (F_{\alpha}{}^{\nu}{}_{,\mu} + F^{\nu}{}_{\mu,\alpha} + F_{\mu\alpha,}{}^{\nu}) \\ &= 0. && \text{(by Maxwell's second equation)} \end{aligned}$$

□

Remark 3.7. Note that $T^{\mu\nu}$ is trace free, so one may also use $(T^{\mu\nu} T_{\mu\nu})_{,\mu} = 0$ to show the proposition.

So far Maxwell's equation get simplify via unification, we have seen

- Lorentz's unification of spacetime,
- Maxwell's unification of electric-magnetic field,
- It remains unknown if we may unify: vacuum energy, Higgs boson and quantum gravity into this picture.

4 Hodge-Maxwell Theorem

We may formulate Maxwell's equation using Hodge theory by identifying E and B as two forms (here I use Einstein summation). For sake of time, I'll merely list the results.

$$\begin{aligned}
E &= E_i dx^i, \\
B &= B_1 dx^2 \wedge dx^3 + B_2 dx^3 \wedge dx^1 + B_3 dx^1 \wedge dx^2 \\
J &= J_\mu dx^\mu,
\end{aligned}$$

Let $F = E \wedge dx^0 + B$ be the Faraday 2-form. One can check $B = *(dx^0 \wedge E)$ and $dx^0 \wedge E = - * B$. [2] We have,

$$G^{\mu\nu}{}_{,\nu} = 0 \iff dF = 0. \tag{M**}$$

$$F^{\mu\nu}{}_{,\nu} = \mu_0 J^\nu \iff d * F = *\mu_0 J, \tag{E**}$$

and the energy-momentum tensor is given by the same formula in Definition 3.4.

5 In curved spacetime

The benefit of having tensor calculus and hodge star operator is we can generalize Maxwell's equations to a curved spacetime. As suggested earlier, we do this by perturbation $g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu}$. Then $F^{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ (in Minkowski space) is replaced by $F^{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ [4, Section 3] and Maxwell's equations become

homogeneous	inhomogeneous
$\nabla_{[\alpha} F_{\beta\gamma]} = 0$	$\nabla_\mu F^{\mu\nu} = \mu_0 J^\nu$

Aside, the process of replacing the partial derivatives with covariant derivatives is similar to how we generalize the Laplace type operator. From a Mathematical perspective, it might be more useful to look at the later case.

[2] I am not going to specify what Hodge star means. One can find detail in [1, Section 3] or [2]

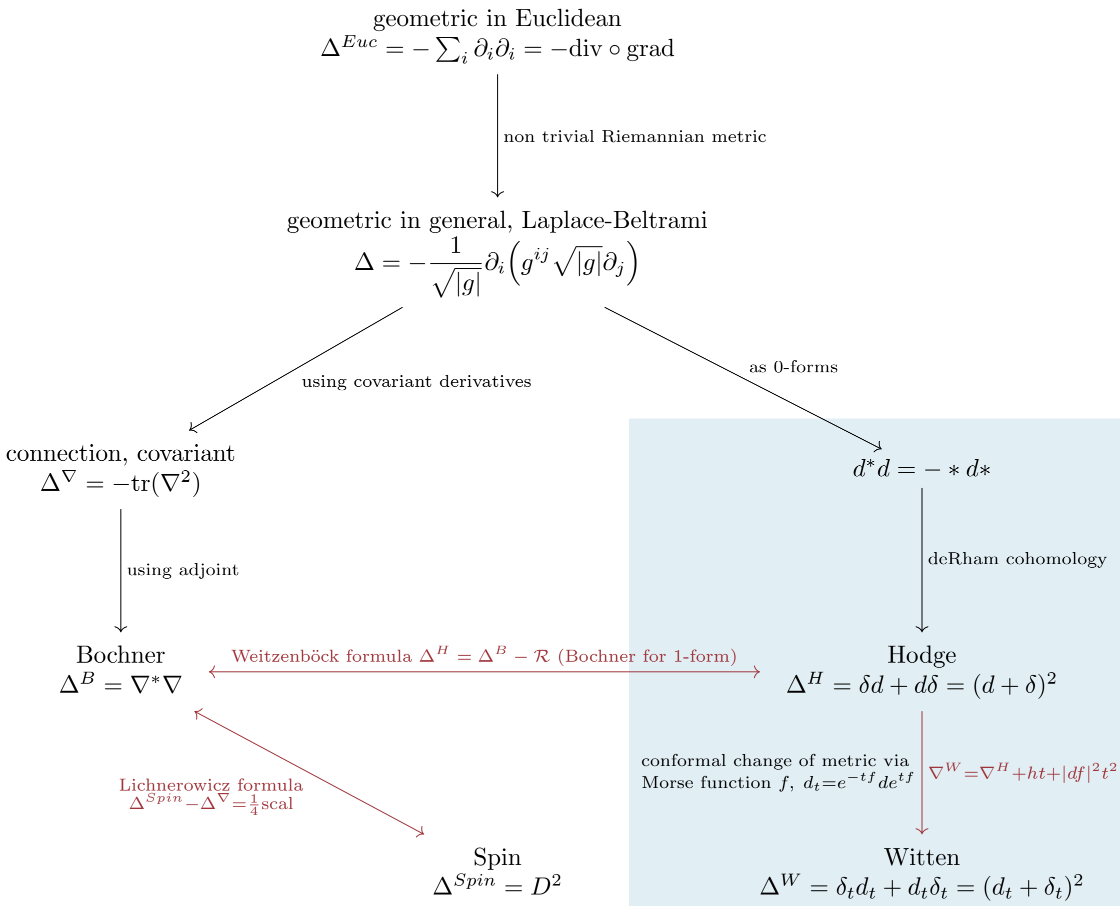

Figure 3: Generalized laplacian

6 Einstein field equation

We now have the terminology needed for the Einstein field equation. This equation generalizes Newton's law of universal gravitation.

$$F = \frac{GMm}{r^2}.$$

Newton's law fails to predict the behavior of photons, since they have no mass. Due to time limitation of the talk, I will write down the equation without a proof.

$$\mathrm{Ric}_{\mu\nu} - \frac{1}{2}\mathrm{scal}\, g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}.$$

Here

- G is the gravitational constant
- $T_{\mu\nu}$ energy-momentum tensor which is extremely large
- $\dfrac{8\pi G}{c^4} = 2.068 \times 10^{-43} \,\dfrac{\mathrm{sec}^2}{\mathrm{kg} \cdot \mathrm{m}}$

Note that the left hand side consists of curvature and metric, and the right hand side consists of gravity, energy and momentum. We may interpret this equation as gravity curves the spacetime; curved spacetime tells how particles behave in gravitation field.

Remark 6.1.

1. The Einstein-Maxwell equations of gravitation and electromagnetism consist of the Einstein field equation with $\nabla_\mu T^{\mu\nu} = 0$.
2. One may add the effect of cosmological constant and obtain generalized version of the above equation

   $$\mathrm{Ric}_{\mu\nu} - \frac{1}{2}\mathrm{scal}\, g_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}.$$

   This extra term won't effect $\nabla^\mu T_{\mu\nu}$ since $\nabla^\mu g_{\mu\nu} = 0$.

Note that we have Einstein condition when the right hand side vanishes. This says Minkowski metric is a trivial solution of the Einstein field equation. There are also non trivial solutions: the Schwarzschild and Kerr metrics [3, Section 6].

Remark 6.2.

1. Non trivial solutions contains singularities at a finite distance from the source. These are called event horizons.

2. At infinity, singularities are normalized, so spacetime looks flat far away and Maxwell field $T_{\mu\nu}$ dominates the behavior.

References

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