The Ambient Obstruction Tensor

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Feb 3, 2023

Abstract

The ambient obstruction tensor is a higher even-dimensional generalization of the Bach tensor. Analogous to the Bach tensor, the obstruction tensor arises from the first variation of a particular conformal invariant, the integral of Branson's Q-curvature. It inherits interesting properties such as conformal invariance and vanishing for conformal Einstein metrics. From another point of view, this tensor obstructs the existence of a smooth power series solution for a Poincaré metric, hence the name ambient obstruction. In this talk, I will go through the later formulation of the obstruction tensor, its basic properties, and its link to the Q-curvature.

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1 Introduction

In 4 dimensional conformal geometry, the Weyl tensor W is a conformal invariant, and the corresponding integral $\int |W|^2$, gives a global conformal invariant. First variation on the metric g gives rise to the so-called Bach tensor:

$$\int |W|^2_{g(t)}\, d\mu_{g(t)} = \int |W|^2_g\, d\mu_g + t \int \langle B, g' \rangle_g\, d\mu_g + O(t^2).$$

In local coordinates,

$$B_{ij} = P^{kl} W_{ikjl} + \nabla^k \nabla_k P_{ij} - \nabla^k \nabla_i P_{jk}.$$

One can check that the Bach tensor is a trace free, symmetric conformally invariant 2-tensor (see Table 1, left).

A higher even-dimensional generalization of the Bach tensor B is the ambient obstruction tensor $\mathcal{O}$. Analogously, $\mathcal{O}$ arise from first variation of Q-curvatures. Another formulation, which motivates where the name "obstruction" comes from, suggests that $\mathcal{O}$ obstructs the existence of a smooth power series solution for the ambient metric associated to a given conformal structure. This approach leads to the Fefferman-Graham expansion, which can be used to construct renormalized volume [3]. We will follow the later formulation.

Bach tensor B_{ij}	ambient obstruction tensor $\mathcal{O}_{ij}$
4-dim	n-dim, $n \geq 4$ even
first variation of $\int |W|^2\, d\mu$	first variation of $\int Q\, d\mu$
conformally invariant	conformally invariant
trace-free	trace-free
symmetric 2 tensor	symmetric 2 tensor
vanishes for conformally Einstein metrics	vanishes for conformally Einstein metrics
involving 4 derivatives of the metric tensor	involving n derivatives of the metric tensor

Table 1: Properties of Bach and obstruction tensors

Let us fix the following notations.

Conformal structure

- M - manifold with smooth conformal structure $[h]$, $\dim M = n \geq 3$.
- X - manifold with boundary M, $\dim X = n + 1$.

- g - conformally compact metrics on X with conformal infinity $[h]$.

  x - smooth boundary defining function, $x^2 g$ smooth on X with $x^2 g|_{TM} \in [h]$

$$g = \frac{1}{x^2}(dx^2 + h_x).$$

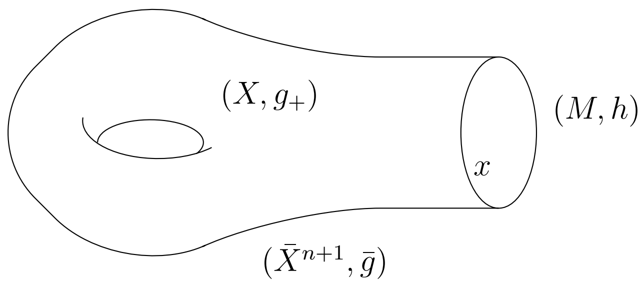

Figure 1: Manifold with boundary, bdf and conformal infinity

2 The ambient obstruction tensor

In this section, we will solve the equation $\mathrm{Ric}_g = -ng$ using formal power series expansion. The main result is the following theorem.

Theorem 2.1 ([4]).

1. *If* $n \geq 4$ *even, there is a metric* g *with*
   - $x^2 g$ *smooth*
   - $[h]$ *is its conformal infinity*
   - $\mathrm{Ric}_g + ng = O(x^{n-2})$.

   This metric g *is unique* $\mod O(x^{n-2})$ *up to a diffeomorphism* ϕ *of* X *with* $\phi|_M = \mathrm{id}$.

2. *We define the obstruction tensor*

$$O = c_n \mathrm{tf}\Big(x^{2-n}(\mathrm{Ric}_g + ng)\Big)\Big|_{TM}, \quad c_n = \frac{2^{n-2}(n/2-1)!^2}{n-2}.$$

   The tensor $\mathcal{O}$ *is well defined: it is independent of the choice of* g *on* M. *Furthermore,*

(a) $\mathcal{O}_{ij} = \Delta^{n/2-2}(P_{ij,k}{}^{k} - P_{k}{}^{k}{}_{ij}) + \text{l.o.t}$

(b) $\mathcal{O}_{i}^{\ i} = 0, \mathcal{O}_{ij,}{}^{j} = 0$

(c) $\mathcal{O}_{ij}$ *is conformally invariant of weight* $2-n$*, i.e.* $\tilde{\mathcal{O}}_{ij} = e^{(2-n)f}\mathcal{O}_{ij}$ *when* $\tilde{h}_{ij} = e^{2f}h_{ij}$.

(d) $\mathcal{O}_{ij} = 0$ *for metrics that are conformal to Einstein metric.*

2.1 Power series solution of $\mathrm{Ric}_g = -ng$

Given a conformally compact metric g which is also asymptotically Einstein, meaning that

$$\mathrm{Ric}_g + ng = O(x^{-1}),$$

where $g = \dfrac{1}{x^2}(dx^2 + h)$ in a collar neighborhood of M.

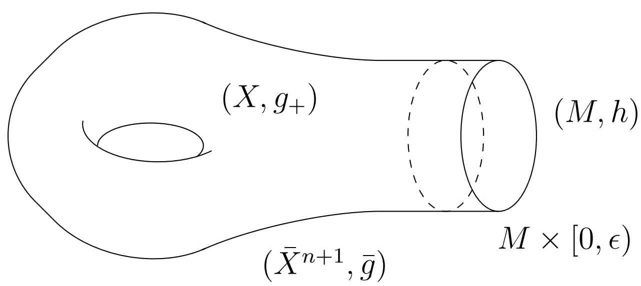

Figure 2: Collar neighborhood

Denote $E = \mathrm{Ric}_g + ng$. Assume there is a formal power series solution

$$h = h_0 + h_1 x + h_2 x^2 + \cdots$$

to the asymptotic equation $E = O(x^{-1})$.

To determine the coefficents h_i, we represent E in terms of the boundary metric h. Recall the Riemannian curvature tensor R is given by Christoffel symbols[1]

$$R_{\alpha\beta\gamma}{}^{\delta} = \partial_\beta \Gamma_{\alpha\gamma}{}^{\delta} - \partial_\alpha \Gamma_{\beta\gamma}{}^{\delta} + \Gamma_{\alpha\gamma}{}^{\mu}\Gamma_{\beta\mu}{}^{\delta} - \Gamma_{\beta\gamma}{}^{\mu}\Gamma_{\alpha\mu}{}^{\delta}.$$

[1]We use Greek letter for indices $0, \cdots n$, Latin letters for indices $1, \cdots, n$.

Computing the Chirstoffel symbol of g in terms of the Chirstoffel symbol of h [1, Lemma 2.1] and substituting into E gives

$$2xE_{ij} = -xh''_{ij} + xh^{kl}h'_{ik}h'_{jl} - \frac{x}{2}h^{kl}h'_{kl}h'_{ij} + (n-1)h'_{ij} + h^{kl}h'_{kl}h_{ij} + 2x\mathrm{Ric}_h \tag{1}$$

$$E_{i0} = \frac{1}{2}h^{kl}(\nabla_l h'_{ik} - \nabla_i h'_{kl}) \tag{2}$$

$$E_{00} = -\frac{1}{2}h^{kl}h'_{kl} + \frac{1}{4}h^{kl}h^{pq}h'_{kp}h'_{lq} + \frac{1}{2x}h^{kl}h'_{kl} \tag{3}$$

We solve $E = O(x^{n-2})$ by induction.

Step 0. Beginning with an initial solution $h_0 = h$.

Step 1. Assume we know h to the $(s-1)$-th order and solve for h_s. Differentiating Equation (1) $s-1$ times results the equation

$$\partial_x^{s-1}\big|_{x=0}(2xE_{ij}) = (n-s)\partial_x^s h_{ij} + h^{kl}\partial_x^s h_{kl}h_{ij} + \text{l.o.t.}$$

Knowing LHS, we may solve for h_s. Indeed, since the operator

$$\begin{aligned}\mathrm{Sym}^2(TM) &\to \mathrm{Sym}^2(TM)\\ \eta_{ij} &\mapsto (n-s)\eta_{ij} + h^{kl}\eta_{kl}h_{ij}\end{aligned}$$

is invertible when s is away from $n, 2n$. This completes the inductive step.

Remark 2.2. The induction ends at $s = n$, so we may solve $h \mod O(x^{n-2})$ by requiring $E_{ij} = O(x^{n-2})$. One may check $E_{i0} = O(x^{n-1})$ and $E_{00} = O(x^{n-2})$ via Bianchi identity and induction. This gives a formal solution to $(n-2)$-th order.

2.2 Properties of $\mathcal{O}_{ij}$

Checking part 2(b)-2(d) of Theorem 2.1 is straightforward (see [4, Theorem 2.1] for detail). We now focus on computing the principle part of $\mathcal{O}_{ij}$. By definition, $\mathcal{O}_{ij}$ corresponds to the coefficient for x^{n-2} in E_{ij}. So restricting $(n-1)$-th derivative of $2xE_{ij}$ to boundary gives the answer.

Remark 2.3.

1. $\mathcal{O}_{ij}$ lives on the boundary. So we differentiate $2xE_{ij}$ instead of E_{ij} in order to avoid blow up when restricting to $x = 0$.

2. Parity: setting $x = 0$ for Equation (1) leads to vanishing of $\partial_x h\big|_{x=0}$. Induction gives $\partial_x^s h\big|_{x=0} = 0$ for odd s.

3. Computing $\partial_x^2 h\big|_{x=0} = -P_{ij}$ is straightforward.

4. Covariant derivative comes from

$$\begin{aligned}\partial_x \mathrm{Ric}_{ij} &= \frac{1}{2}\partial_x(h_{ik,j}{}^{k} + h_{jk,i}{}^{k} - h_{ij,k}{}^{k} - h_{k}{}^{k}{}_{,ij}) \\ \implies \partial_x \mathrm{Ric}\big|_{x=0} &= \frac{1}{2}\Delta(h_2) - \delta^*\delta(h_2) - \frac{1}{2}\mathrm{Hess}(\mathrm{tr}(h_2)).\end{aligned}$$

Example 2.4. [4]

- For $n = 4$, $\mathcal{O}_{ij} = B_{ij}$.
- For $n = 6$,

$$\begin{aligned}\mathcal{O}_{ij} =& B_{ij,k}{}^{k} - 2W_{kijl}B^{kl} - 4P_{k}{}^{k}B_{ij} + 8P^{kl}C_{(ij)k,l} \\ &- 4C^{k}{}_{i}{}^{l}C_{ljk} + 2C_{i}{}^{kl}C_{jkl} + 4P^{k}{}_{k,l}C_{(ij)}{}^{l} - 4W_{kijl}P^{k}{}_{m}P^{ml}.\end{aligned}$$

3 Link to Q-curvature

Remark 3.1.

1. Q itself is not a pointwise conformal invariant, but its integral is.
2. $\mathcal{O}_{ij}$ obstructs a smooth formal power series solutions for a Poincaré metric (linking Q to $\mathcal{O}$).

$$\begin{array}{ccccc} \text{volume expansion of Poincaré metric} & \longrightarrow & \text{log term} & \longleftarrow & \int Q\, d\mu \\ & & \big\downarrow{\scriptstyle \text{variation}} & & \\ & & \text{obstruction tensor} & & \end{array}$$

The above construction is called the Fefferman-Graham expansion. It also works for odd dimension and there is no obstruction at $(n-2)$-th order:

$$h_x = \begin{cases} h_0 + h_2\rho^2 + (\text{even powers}) + h_{n-1}\rho^{n-1} + h_n\rho^n + \cdots & n \text{ odd} \\ h_0 + h_2\rho^2 + (\text{even powers}) + h_{n,1}\log(\rho)\rho^{n-1} + h_n\rho^n + \cdots & n \text{ even.} \end{cases}$$

This implies a power series expansion for the volume form thus for the volume [3]. For n even,

$$\mathrm{Vol}_g(\{x > \epsilon\}) = c_0\epsilon^{-n} + c_2\epsilon^{-n+2} + \text{ (even powers) } + c_{n-2}\epsilon^{-2} + L\log\frac{1}{\epsilon} + V + o(1),$$

where $L = \int_M v^{(n)}\, d\mu_h$ and $v^{(n)}$ is the coefficient for x^{-1} of the volume form, is a conformal invariant.

Theorem 3.2. *Let $h(t)$ be a 1-parameter family of metrics on a compact manifold M of even dimension $n \geq 4$, then*

$$\frac{\partial}{\partial t}\Big|_{t=0} \int_M Q\, d\mu = (-1)^{n/2}\, \frac{n-2}{2} \int_M \mathcal{O}_{ij}\, \frac{\partial}{\partial t}\Big|_{t=0} h^{ij}\, d\mu$$

Recall that the leading order term for $\mathcal{O}$ is $\Delta^{n/2-2}(P_{ij,k}{}^{k} - P_{k}{}^{k}{}_{ij})$. The fact that $Q = -\dfrac{1}{2(2n-1)}\Delta^{n-1}R + \text{l.o.t}$ [2] would convince you of the above theorem.

Example 3.3. In dimension 4, the Q-curvature is $\frac{1}{6}(-\Delta R + R^2 - 3|\mathrm{Ric}|^2)$. Chern-Gauss-Bonnet says

$$\chi(M) = \frac{1}{32\pi^2} \int_M (|\mathrm{Rm}|^2 - 4|\mathrm{Ric}|^2 + R^2)\, d\mu \implies \int_M Q\, d\mu = 8\pi^2\chi(M) - \frac{1}{4}\int_M |W|^2\, d\mu.$$

So the obstruction tensor is the Bach tensor. •

References

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