The Fractional Laplacian Through Dirichlet Problem Formulation

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Abstract

The fractional Laplacian is a generalization of the classical Laplacian operator to non-integer orders. It was first introduced in Grušin's 1960 work on spectral theory. This operator can be expressed through various formulations, including Fourier transforms and extensions of the Dirichlet problem for the Poisson equation to a conformally covariant boundary value problem. Notably, the latter formulation contributes to the theory of Sobolev spaces, elucidating a precise higher-order Sobolev trace inequality.

Contents

1 Introduction

These notes seek to provide a introductory understanding of the fractional Laplacian through multiple approaches. First, we will explore a pointwise definition of the fractional Laplacian. Then, we will focus on the Fourier transform, which tells differential operators act on functions as integrals, thus offers a more intuitive extension of the classical Laplacian using principal symbols. Finally, we will delve into an equivalent formulation involving an extension problem within the framework of Poisson operators and scattering theory, offering an alternative perspective on the fractional Laplacian. There are more ways to define the fractional Laplacian, Kwaśnicki [Kwa17] has a nice survey discussing those.

Throughout the notes we will be dealing with the Schwartz functions. We recall that the Schwartz functions are rapidly decreasing functions such that

$$\mathscr{S}(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) : \|f\|_{\alpha,\beta} = \sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta f(x)| < \infty, \forall \alpha, \beta\}.$$

It is easy to check that $C_c^\infty(\mathbb{R}^n) \subset \mathscr{S}(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n)$.

We introduce the first definition of the fractional Laplacian. Note that the classical Laplacian can be written in terms of averaging integral

$$\begin{aligned} -f''(x) &= \lim_{y\to 0} \frac{2f(x) - f(x+y) - f(x-y)}{y^2} \\ &= 6 \lim_{y\to 0} \frac{f(x) - \mathscr{A}_y f(x)}{y^2}, \qquad \text{where } \mathscr{A}_y f(x) = \frac{1}{2y}\int_{x-y}^{x+y} f(t)dt. \end{aligned}$$

In n-dimension, one uses $\mathscr{A}_y f(x) = \frac{1}{\omega_n r^n}\int_{B(x,r)} f(y)dy$ to get

$$-\Delta f(x) = 2(n+2)\lim_{y\to 0} \frac{f(x) - \mathscr{A}_y f(x)}{y^2}.$$

This leads to the first pointwise definition. [1]

Definition 1.1 (Pointwise definition)**.** For $\gamma \in (0,1)$,

$$(-\Delta)^\gamma f(x) = \frac{c(n,\gamma)}{2}\int \frac{2f(x) - f(x+y) - f(x-y)}{|y|^{n+2\gamma}} dy.$$

[1] Minus sign is needed since Δ denotes the analyst Laplacian.

Remark 1.2.

1. As γ tends to 1, the operator approaches the classical Laplacian. It is anticipated that $(-\Delta)^\gamma$ exhibits behaviors closely resembling $-\Delta$. Similarly, as γ tends to 0, the operator approaches the identity map. Consequently, one expects $(-\Delta)^\gamma$ to display behaviors akin to those of the identity operator id.

2. This can also be written as a single integral. For $\gamma \in (0,1)$,

$$(-\Delta)^\gamma f(x) = C_{n,\gamma} P.V. \int_{\mathbb{R}^n} \frac{f(x) - f(\xi)}{|x-\xi|^{n+2\gamma}} d\xi.$$[2]

2 Fourier transform

Recall that the Fourier transform $\mathcal{F} : L^1(\mathbb{R}^m) \to L^\infty(\mathbb{R}^n)$

$$\mathcal{F}(f)(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{-i\xi x} dx$$

has the following properties on Schwartz space.

Proposition 2.1.

1. For $f \in \mathscr{S}(\mathbb{R}^m)$,

$$\mathcal{F}(\partial_{x_j} f)(\xi) = i\xi_j \mathcal{F}(f)(\xi), \quad \partial_{\xi_j}\big(\mathcal{F}(f)\big)(\xi) = -i\mathcal{F}(\xi_j f)(\xi);$$

2. On Schwartz space Fourier transform $\mathcal{F} : \mathscr{S}(\mathbb{R}^n_x) \to \mathscr{S}(\mathbb{R}^n_\xi)$ has a formal adjoint $\mathcal{F}^* : \mathscr{S}(\mathbb{R}^n_\xi) \to \mathscr{S}(\mathbb{R}^n_x)$

$$\mathcal{F}^*(f)(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{i\xi x} dx.$$

Moreover, $\mathcal{F}^{-1} = \mathcal{F}^*$ on the Schwartz functions.

[2] P.V. stands for Cauchy principal value. $P.V. \int_{\mathbb{R}^n} u\, dx = \lim_{\epsilon\to 0} \int_{\mathbb{R}^n \setminus B_\epsilon(0)} u\, dx$.

A key aspect of the Fourier transform is that when you apply a differential operator to a function in the spatial domain $\mathbb{R}^n_x$, it corresponds to multiplying the Fourier transform of that function by a certain polynomial in the frequency domain $\mathbb{R}^n_\xi$. This operation can be represented mathematically as convolution with an *integral kernel* $\mathcal{K}_P$,

$$P(f)(x) = \int_{\mathbb{R}^n} \mathcal{K}_P(x,y) f(y) dy.$$

Let's make this more concrete. For a differential operator P of the form $\sum_{\alpha \le k} a_\alpha(x) D^\alpha$, where $D_a := \frac{1}{i}\partial_a$, we have

$$\begin{aligned} P(f)(x) &= \mathcal{F}^*\big(\sigma_P(x,\xi)\mathcal{F}f\big)(x) = \frac{1}{(2\pi)^n} \iint e^{i(x-y)\cdot\xi} \sigma_P(x,\xi) f(y)\, dy\, d\xi \\ &= \int \underbrace{\frac{1}{(2\pi)^n} \int e^{i(x-y)\cdot\xi} \sigma_P(x,\xi) d\xi}_{\mathcal{K}_P(x,y)} \; f(y)\, dy \end{aligned}$$

$$\begin{array}{ccc} \mathscr{S}(\mathbb{R}^n_x) & \xrightarrow{P} & \mathscr{S}(\mathbb{R}^n_x) \\ \Big\downarrow \mathcal{F} & & \mathcal{F}^* \Big\uparrow \\ \mathscr{S}(\mathbb{R}^n_\xi) & \xrightarrow{\sigma(P)} & \mathscr{S}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi) \end{array} \qquad \begin{array}{ccc} f & \xmapsto{\quad P \quad} & P(f) \\ \Big\downarrow \mathcal{F} & & \mathcal{F}^* \Big\uparrow \\ \mathcal{F}(f) & \xmapsto{\sigma(P)} & \sigma(P)(x,\xi)\mathcal{F}(f) \end{array}$$

In particular, for a Laplace type operator, we have

$$\mathcal{F}\big((-\Delta)f\big)(\xi) = |\xi|^2 \mathcal{F}(f)(\xi).$$

This leads to the second definition of the fractional Laplacian.

Definition 2.2 (by Fourier transform)**.** We define $(-\Delta)^\gamma$ to be the differential operator whose principal symbol is $|\xi|^{2\gamma}$, i.e.

$$\mathcal{F}\big((-\Delta)^\gamma f\big)(\xi) = |\xi|^{2\gamma} \mathcal{F}(f)(\xi).$$

3 Caffarelli-Silvestre extension problem

As it mentioned in the introduction, we may formulate an extension problem and recover the fractional Laplacian from it. Consider $\mathbb{R}^{n+1}_+$ with local coordinates $(x,y) \in$

$\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times \mathbb{R}_+$. Let $\gamma \in (0,1)$. The Caffarelli-Silvestre gives a interpretation of the fractional Laplacian using Dirichlet-to-Neumann map associated to the weighted Laplacian [CY23].

Definition 3.1 (by Caffarelli-Silvestre extension theorem)**.** If U is a solution to the weighted Laplacian $\Delta_m := \Delta_x + m\frac{1}{y}\partial_y + \partial_y^2$, $m = 1 - 2\gamma$. That is,

$$\begin{cases} \Delta_m U(x,y) = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ U(x,0) = f(x) & \text{on } \mathbb{R}^n. \end{cases} \tag{1}$$

Then the Dirichlet-to-Neumann operator $f \to \partial_y U$ recovers $(-\Delta_x)^\gamma$. We have

$$(-\Delta_x)^\gamma f = -2^{2\gamma-1}\frac{\Gamma(\gamma)}{\Gamma(1-\gamma)}\lim_{y\to 0} y^{1-2\gamma}\,\partial_y U. \tag{2}$$

Remark 3.2. The Poisson kernel for $(-\Delta)^\gamma$ is $K_\gamma(x,y) = c_{n,\gamma}\frac{y^{2\gamma}}{\left(|x|^2+|y|^2\right)^{\frac{n}{2}+\gamma}}$ and $U = K_\gamma *_x f$. One may compare this with Definition 2.

A particular case is when $\gamma = \frac{1}{2}$, $\Delta^{1/2}$ is the Dirichlet-to-Neumann map

$$f \mapsto -U_y(x,0).$$

Moreover, the Dirichlet principle provide a sharp trace inequality.

Theorem 3.3 (Dirichlet's principle)**.** *Suppose* $U \in C^1(\Omega) \cup C^0(\overline{\Omega})$ *is the solution to Poisson's equation*

$$\begin{cases} \Delta U + g = 0 & \text{in } \Omega \\ U = f & \text{on } \partial\Omega \end{cases}$$

then u *can be obtained as the minimizer of the Dirichlet energy*

$$\mathcal{E}[V] = \int_\Omega \frac{1}{2}|\nabla V|^2 - Vg\,dx$$

where $V \in C^1(\Omega) \cup C^0(\overline{\Omega})$ *such that* $v = f$ *on* $\partial\Omega$.

Theorem 3.4 (Sharp trace inequality)**.** *For $\gamma \in (0,1)$,*

$$2^{1-2\gamma}\frac{\Gamma(1-\gamma)}{\Gamma(\gamma)}\int_{\mathbb{R}^n}(-\Delta_x)^\gamma f\,dx \le \int_{\mathbb{R}^{n+1}_+}|\nabla U|^2 y^{1-2\gamma}\,dx\,dy,$$

with equality holds precisely when U solves the extension problem (1).

4 Higher order fractional Laplacian

In this section, we'll explore how we can expand the applicability of the fractional Laplacian by defining it for any $\gamma \in (0,\infty)\backslash\mathbb{N}$, as well as for particular types of manifolds with boundaries. In the realm of real analysis, this extension incorporates Poincaré-Einstein manifolds, whereas in the complex domain, it encompasses asymptotically complex hyperbolic Einstein manifolds.

4.1 Scattering formulation

Chang and González [CdMG10] extended the fractional Laplacians to higher order through Graham-Zworski scattering theory. This process gives a clean formula of $(-\Delta)^\gamma$ via taking inductive derivatives. However, it does not provide a trace inequality.

Given $\gamma \in (0,\frac{n}{2})\backslash\mathbb{N}$, for a function $f \in C^\infty(\mathbb{R}^n)$ we consider the boundary value problem

$$\begin{cases}\Delta_x U + (1-2\gamma)\frac{1}{y}\partial_y U + \partial_y^2 U = 0 & \text{in } \mathbb{R}^{n+1}_+\\ U(x,0) = f(x) & \text{on } \mathbb{R}^n.\end{cases}$$

Remark 4.1. Note that the weighted Laplacian can be formulate using 0-calculus. When the boundary fibration is given by $\{\text{pt}\} - \partial X \hookrightarrow M$, the Laplacian is of the form $-\Delta_x - v\frac{1}{y}\partial_y - +\partial_y^2$.

Denote U to be a solution to the boundary value problem, then $u = y^{n-s}U$ is a solution to the following Poisson problem

$$\begin{cases}-\Delta_{g_{\mathbb{H}^{n+1}}} u - s(n-s)u = 0 & \text{in } \mathbb{R}^{n+1}_+\\ u = y^{n-s}F + y^s G & \\ F(x,0) = f(x) & \text{on } \mathbb{R}^n.\end{cases}$$

Chang and González used the Poisson problem to recover fractional Laplacian

$$(-\Delta)^\gamma f = \frac{d_\gamma}{2[\gamma]} A_\gamma^{-1} \lim_{y\to 0} y^{1-2[\gamma]} \partial_y \Big(\frac{1}{y}\partial_y\Big)^{\lfloor\gamma\rfloor} U,$$

where

$$d_\gamma = 2^\gamma \frac{\Gamma(\gamma)}{\Gamma(-\gamma)}, \quad A_\gamma = 2^{\lfloor\gamma\rfloor}(\gamma-1)\cdots(\gamma-\lfloor\gamma\rfloor+1).$$

In particular, when $\gamma \in (0,1)$ we recover (2).

Remark 4.2. The construction described above applies to Poincaré-Einstein manifolds. It requires understanding the asymptotic behavior of the interior metric $g = \frac{d\rho^2+h}{\rho^2}$ near the boundary in order to accurately formulate the extension problem.

4.2 Boundary operators formulation

In Case's work [Cas21], it has been demonstrated that the corresponding Caffarelli-Silvestre extension problem, along with the trace inequality, holds when suitable higher-order boundary operators are introduced. To better understand the extension problem in this setting, let's first recall the scattering theory. We will be working on a Poincare-Einstein manifold X with boundary $M = \{\rho = 0\}$.

Let Δ_+ be the Laplace-Beltrami operator for the interior metric g_+, and suppose $\frac{n^2}{4} - \gamma^2$ is not in the L^2-spectrum of $-\Delta_+$. Given $f \in C^\infty(M)$ and set $s = \frac{n}{2} + \gamma$. Then there is a unique solution $u = \mathcal{P}(s)f$ of the Poisson equation

$$\Delta_+ u + s(n-s)u = 0,$$

such that near M there is some $F, G \in C^\infty(\bar{X})$,

$$\begin{cases} \mathcal{P}(s)f = \rho^{n-s}F + \rho^s G, \\ F|_M = f. \end{cases}$$

We call the map $\mathcal{P} : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^{n+1}_+); f \mapsto u$ the *Poisson map*, and define the *scattering operator* to be $\mathcal{S}(s)(f) = G|_{\rho=0}$.

Remark 4.3. The *GJMS operator* of order 2γ is the normalized scattering operator [GJMS92, FdMGMT15]:

$$P_\gamma^\theta = 2^\gamma \frac{\Gamma(\gamma)}{\Gamma(-\gamma)} \mathcal{S}\Big(\frac{m+\gamma}{2}\Big).$$

One can check directly that the GJMS operator is conformal covariant in the sense that for smooth w, [CY23]

$$P_\gamma^{e^{2w}\theta} f = e^{-(m+\gamma)w} P_\gamma^\theta \Big(e^{(m-\gamma)w} f\Big).$$

Let $\gamma \in (0,\infty)\backslash\mathbb{N}$, and decompose $\gamma = \lfloor\gamma\rfloor + [\gamma]$ into integer and fractional part. Set $m = 1 - 2[\gamma]$ and $k = \lfloor\gamma\rfloor + 1$. We define the weighted poly Laplacian to be k-th power of the weighted Laplacian Δ_m, i.e.

$$L_{2k} = \Delta_m^k = \left(\Delta + m\frac{1}{y}\partial_y\right)^k$$

and boundary operators $B_{2j}^{2\gamma}$ on the function space $C^{2\gamma}(\mathbb{R}_+^{n+1})$. Then the generalized extension problem becomes

$$\begin{cases} L_{2k}(V) = 0 & \text{in } \mathbb{R}_+^{n+1} \\ B_{2j}^{2\gamma}(V) = f^{(2j)} & j \in \big[0, \lfloor\frac{\gamma}{2}\rfloor\big] \\ B_{2\gamma-2j}^{2\gamma}(V) = \phi^{(2j)} & j \in \big[0, \lfloor\gamma\rfloor - \lfloor\frac{\gamma}{2}\rfloor - 1\big]. \end{cases}$$

Boundary operators $B_{2j}^{2\gamma}, B_{2[\gamma]+2j}^{2\gamma} : C^{2\gamma}(\mathbb{R}_+^{n+1}) \to C^\infty(\mathbb{R}^n)$ are recursively defined for $0 \le j \le k-1$ as follows

$$\begin{aligned} B_{2j}^{2\gamma} &= (-1)^j \iota^* \circ \left(\partial_y^2 + m\frac{1}{y}\partial_y\right)^j \\ &\quad - \sum_{\ell=1}^{j} \binom{j}{\ell} \frac{\Gamma(1+j-[\gamma])\,\Gamma(1+2j-2\ell-\gamma)}{\Gamma(1+j-\ell-[\gamma])\,\Gamma(1+2j-\ell-\gamma)} \Delta_x^\ell B_{2j-2\ell}^{2\gamma}, \\ B_{2[\gamma]+2j}^{2\gamma} &= (-1)^{j+1} \iota^* \circ y^m \partial_y \Big(\partial_y^2 + m\frac{1}{y}\partial_y\Big)^j \\ &\quad - \sum_{\ell=1}^{j} \binom{j}{\ell} \frac{\Gamma(1+j+[\gamma])\,\Gamma(1+2j-2\ell-\lfloor\gamma\rfloor+[\gamma])}{\Gamma(1+j-\ell+[\gamma])\,\Gamma(1+2j-\ell-\lfloor\gamma\rfloor+[\gamma]} \Delta_x^\ell B_{2[\gamma]+2j-2\ell}^{2\gamma}. \end{aligned}$$

Here $\iota^* : C^\infty(\overline{\mathbb{R}_+^{n+1}}) \to C^\infty(\mathbb{R}^n)$ is the restriction map.

Remark 4.4. Another way to define the boundary operator, is given in [FLY23b]

$$
\begin{aligned}
B^{2\gamma}_{2j} &= b_{2j}\rho^{-\frac{n}{2}+\gamma-2j} \circ \prod_{\ell=0}^{j=1}\left(\Delta_+ + \frac{n^2}{4} - (\gamma-2\ell)^2\right) \\
&\circ \prod_{\ell=0}^{j=1}\left(\Delta_+ + \frac{n^2}{4} - (\gamma-2\ell-2\lfloor\gamma\rfloor)^2\right) \circ \rho^{\frac{n}{2}-\gamma}\Big|_{\rho=0}; \\
B^{2\gamma}_{2j+2[\gamma]} &= b_{2j+2[\gamma]}\rho^{-\frac{n}{2}+\gamma-2j-2[\gamma]} \circ \prod_{\ell=0}^{j=1}\left(\Delta_+ + \frac{n^2}{4} - (\gamma-2\ell)^2\right) \\
&\circ \prod_{\ell=0}^{j=1}\left(\Delta_+ + \frac{n^2}{4} - (\gamma-2\ell-2\lfloor\gamma\rfloor)^2\right) \circ \rho^{\frac{n}{2}-\gamma}\Big|_{\rho=0}.
\end{aligned}
$$

We define the associated Dirichlet form to be

$$
\begin{aligned}
\mathcal{Q}_{2\gamma}(U,V) = &\int_{\mathbb{R}^{n+1}_+} U L_{2k} V y^m \, dxdy \\
&+ \sum_{j=0}^{\lfloor\frac{\gamma}{2}\rfloor} \int_{\mathbb{R}^n} B^{2\gamma}_{2j}(U) B^{2\gamma}_{2\gamma-2j}(V)\, dx - \sum_{j=0}^{\lfloor\gamma\rfloor-\lfloor\frac{\gamma}{2}\rfloor-1} \int_{\mathbb{R}^n} B^{2\gamma}_{2\gamma-2j}(U) B^{2\gamma}_{2j}(V)\, dx.
\end{aligned}
$$

This is a symmetric two form, whose trace is the corresponding Dirichlet energy $\mathcal{E}(U)$.

Theorem 4.5 (Sharp Sobolev trace inequality)**.** *For all function $U \in C^{2\gamma}(\mathbb{R}^{n+1}_+) \cap W^{\lfloor\gamma\rfloor+1,2}(\mathbb{R}^{n+1}_+, y^{1-2[\gamma]})$, $f^{(2j)} = B^{2\gamma}_{2j}(U)$ and $\phi^{(2j)} = B^{2\gamma}_{2[\gamma} + 2j](U)$, where $0 \le j \le \lfloor\gamma\rfloor - \lfloor\frac{\gamma}{2}\rfloor - 1$,*

$$
\mathcal{E}_{2\gamma}(U) \geq \sum_{j=0}^{\lfloor\frac{\gamma}{2}\rfloor} \int_{\mathbb{R}^n} c_{\gamma,j} f^{(2j)}(-\Delta)^{\gamma} f^{(2j)}\, dx + \sum_{j=0}^{\lfloor\gamma\rfloor-\lfloor\frac{\gamma}{2}\rfloor-1} \int_{\mathbb{R}^n} d_{\gamma,j}\phi^{(2j)}(-\Delta)^{\lfloor\gamma\rfloor-[\gamma]-2j}\phi^{(2j)}\, dx.
$$

Moreover, equality holds if and only if $L_{2k}(U) = 0$.

The aforementioned construction is applicable to complex manifolds, as elaborated in [FLY23a, Wan17]. The key idea revolves around formulating the extension problem with suitable boundary operators B, which essentially encode the information from F and G in the scattering problem. These approaches also yields energy inequalities.

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