Simulating and Understanding Multi-Soliton Solutions to NLS

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Introduction

Goals

- Gain Familiarity with the nonlinear Schrödinger (NLS) equation
- Use paper by Biondini and Kovačič to help understand and observe behavior and form of the NLS at nonzero boundary cases
- Fix and change code from Zero Boundary Condition (ZBC) solution to new solution for nonzero Boundary Conditions (NZBC)
- Create animations for multi-soliton solutions for the NLS at NZBC

- Let $q(x,t)$ be a function dependent on x and t. A partial derivative of q is a function which gives the change in q when slightly change one variable while keeping the others fixed. Here we will use subscripts to denote partial derivatives: q_x for the change in q with respect to x (keeping t fixed), and q_t for the change in q with respect to t. Note that we can take the partial deriviative of a function twice, which is denoted q_{xx} or q_{tt}. A Partial Differetial Equation(PDE) is then an equation which includes the partial derivatives of q.
- A linear differential equation is an equation which only has terms with either zero or one factors of either q or a derivative of q. The nonlinear Schrödinger Equation is nonlinear because it includes a factor with $|q|^2$. Linear PDEs are generally easier to solve, but nonlinear equations are more common.
- The nonlinear Schrödinger equation is

$$iq_t + q_{xx} + 2(|q|^2 - q_0^2) = 0$$

where q is a complex valued function dependent on x and t, and q_0 is $|q(x,t)|$ far from the origin. We looked at the case where $q_0 \neq 0$.

$$\begin{array}{ccc} q(x,t=0) & \xrightarrow[\text{Scattering}]{\text{Direct}} & S(z,t=0) \\ & & \Big\downarrow \text{Time Evolution} \\ q(x,t) & \xleftarrow[\text{Scattering}]{\text{Inverse}} & S(z,t) \end{array}$$

- The solution to the PDE is in general, an infinite sum. If we look at the solutions with N terms, then we have an N-solition solution. Each term corresponds to one soliton. Visually, a soliton will be a bump in the graph. For example, Figure 1 has 3 solitons.

Comparing Solutions

After arriving at a solution for the Nonzero Boundary Condition (NZBC) equation, a natural way to inspect and investigate the differences of boundary conditions is to compare terms and see how they are different or similar and what impact the differences might have.

Examining the single soliton case for the NZBC and ZBC we noticed a few things that were important aspects of the NLS equation.

A major difference we noticed was that important quantities included in the calculation of the ZBC solution were different in the NZBC solution.

	ZBC	NZBC
$q(x,t)$	$Ae^{i(V^2-A^2)t-iVx-i\psi}\,\mathrm{sech}(Ax-2AVt-\delta)$	$q_0 - i\sum_{n=1}^{2N} C_n^* e^{2i\theta(x,t,\zeta_n^*)}\,\mu_{-,1,1}(x,t,\zeta_n^*)$
Velocity	V	$\frac{V(A^4+2A^2V^2+V^4+16q_0^2)}{(A^2+V^2)(A^2+V^2-4q_0^2)}$
Important Variables	Norming constant involves a scaling by amplitude A, shift of δ, and phase ψ	• Norming constant C_j lacks the scaling A, as well as differs by phase shift ψ for the stationary soliton case • Critical points $x_{\max}$ and $x_{\min}$ of the graph occur at different points in time

Figures and Animations

Figure 1 shows an example solution of the NLS with ZBC. Other solutions have a similar shape.

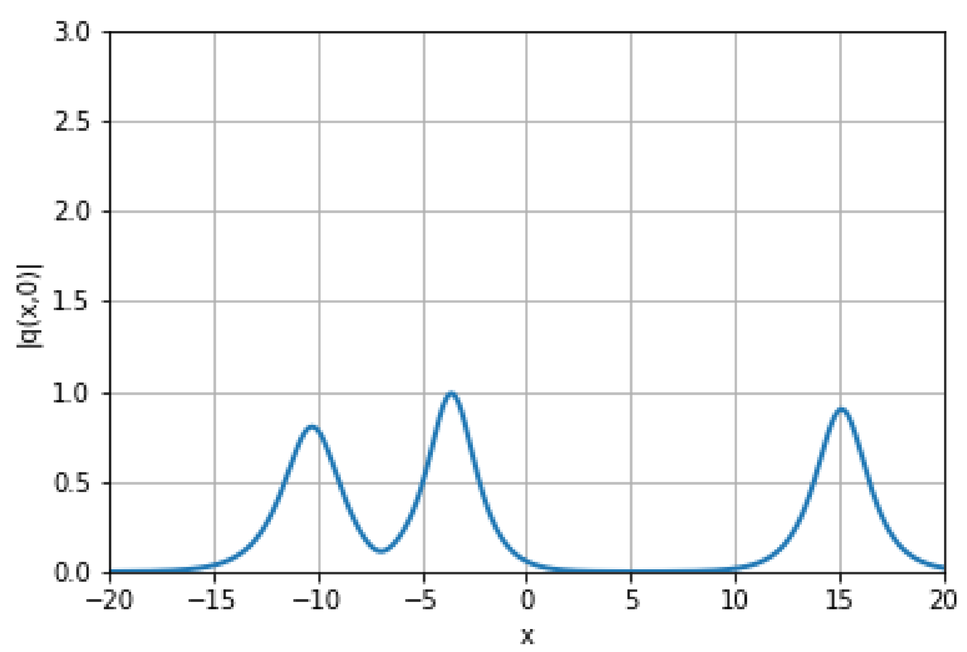

Figure 1: *Example solution for NLS with ZBC*

Stationary Soliton

If the eigenvalue of the solution is purely imaginary, then this solution is stationary. The equation requires that the eigenvalue z cannot be i. Figure 2 shows 3 example soliton solutions.

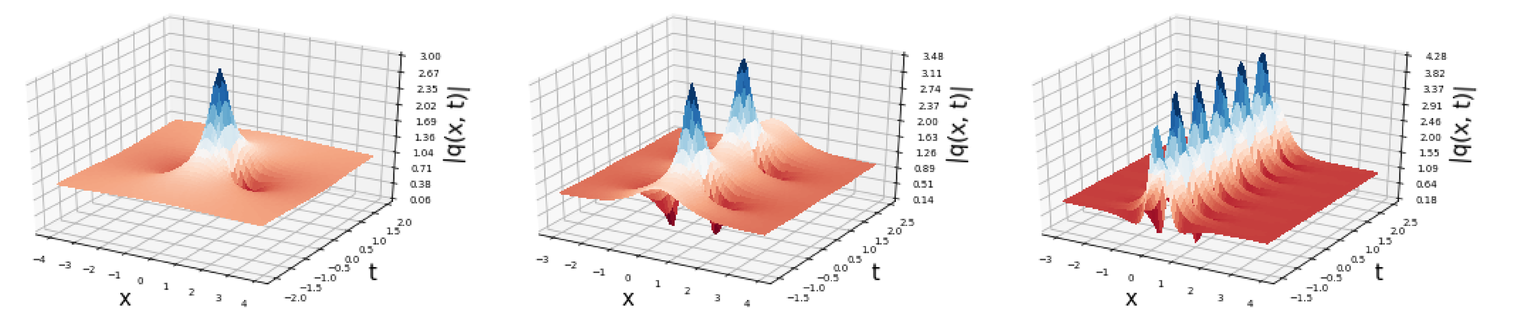

Figure 2: *Three sample stationary solutions of NLS with NZBC. From left to right the eigenvalues are $z \to i$, $z = 2i$ and $z = 3i$. Notice that as we increase the size of the imaginary eigenvalue, we find that there are more peaks over time!*

- In this case, we find that the maximum value of the solution always occurs at $x = 0$, and it is periodic along t-axis.
- If the eigenvalue has a greater modulus, then the solution has a shorter period and greater amplitude.
- When we take the eigenvalue to be very close to i, but not equal to, we will find that there is only one peak, at the origin.

Non-stationary Soliton

If the eigenvalue of the solution is complex ($Ze^{i\alpha}$), then this solution is non-stationary. The following are three examples of non-stationary soliton solutions:

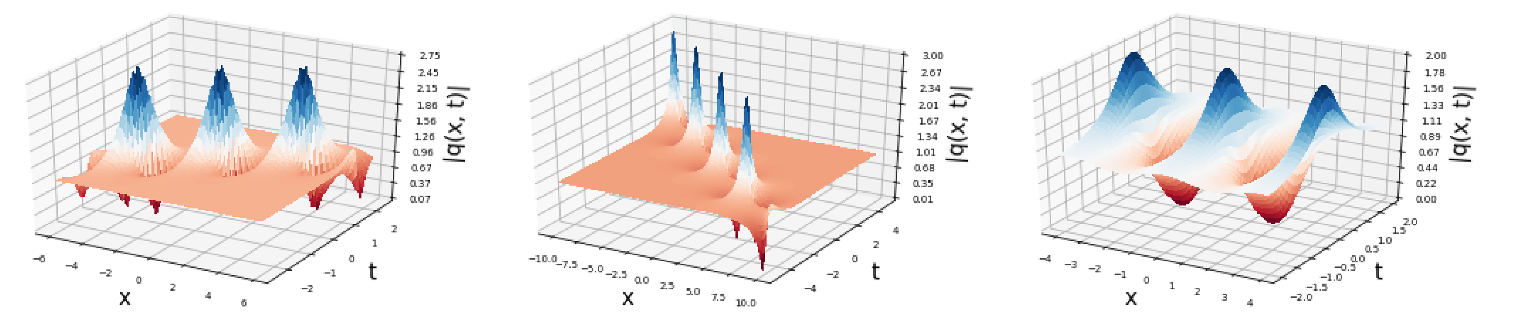

Figure 3: *Three sample non-stationary solutions of NLS with NZBC. Left: $Z = 2$, $\alpha = \pi/4$, middle: $Z = \sqrt{2}$, $\alpha = -\pi/10$, right: $Z \to 1$, $\alpha = \pi/3$.*

- In this case, we find that the solution is periodic along both x-axis and t-axis.
- If $\alpha = 0$, then the solution is stationary.
- Taking $Z \to 1$, the maximum occurs at $t = 0$ and the solution is periodic along the x-axis.
- Notice that in the left and right examples, α was positive, and the peaks appear to travel in the positive x direction as t increases, whereas in the middle figure (with $\alpha < 0$, the behavior is opposite).

Future Directions

There are a few directions to take in the future:

- Animate the solution for a general N-soliton solution.
- Use animations to investigate dispersion relations, which are relations between spacial frequency and temporal frequency.

References

G. Biondini and G. Kovačič. Inverse scattering transform for the focusing nonlinear schrödinger equation with nonzero boundary conditions. *Journal of Mathematical Physics*, 55, 02 2014. doi: 10.1063/1.4868483.

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