# Homework Assignments

MATH231

Spring 2022

# Homework 0

## Due: Friday, Jan 21 (no need to turn in)

1. Calculating Limits

   - $\lim_{x \to 2} \dfrac{x^2 + x - 6}{x - 2}$

   - $\lim_{h \to 0} \dfrac{(x+h)^2 - x^2}{h}$

   - $\lim_{x \to 0} \left( \dfrac{1}{x} - \dfrac{1}{x^2 + x} \right)$

2. The Chain Rule

   - $\dfrac{\mathrm{d}}{\mathrm{d}x} \ln(x + \sin x)$

   - $\dfrac{\mathrm{d}}{\mathrm{d}x} \cos(x^2 e^x)$

3. Implicit Differentiation: Solve for $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ for the following implicit function.

   - $x^2 + y^2 = r^2$, where $r$ is a constant

   - $\dfrac{x + y}{x - y} = x$

4. Linear Approximations and Differentials: Find the Taylor polynomials of degree two approximating the given function centered at the given point.

   - $f(x) = \sin(2x)$ at $a = \frac{\pi}{2}$

   - $f(x) = e^x$ at $a = 1$

5. Mean Value Theorem: Determine if the Mean Value Theorem can be applied to the following function on the the given closed interval.

   - $f(x) = 3 + \sqrt{x}, x \in [0, 4]$

   - $f(x) = \dfrac{x}{1 + x}, x \in [1, 3]$

6. L'Hospital's Rule
   - $\lim_{x \to 2} \frac{x^3 - 7x^2 + 10x}{x^2 + x - 6}$
   - $\lim_{x \to \infty} (e^x + x)^{\frac{1}{x}}$
   - $\lim_{x \to \infty} x \ln\left(1 + \frac{3}{x}\right)$
7. The Fundamental Theorem of Calculus: Find the derivative of the following
   - $\int_1^x \frac{1}{t^3 + 1} \, \mathrm{d}t$
   - $\int_1^{\sqrt{x}} \sin t \, \mathrm{d}t$
   - $\int_x^{2x} t^3 \, \mathrm{d}t$
8. Substitution Rule
   - $\int_{\frac{1}{2}}^{0} \frac{x}{\sqrt{1 - 4x^2}} \, \mathrm{d}x$
   - $\int_{\frac{1}{4}}^{\frac{1}{2}} \frac{\cos(\pi x)}{\sin^2(\pi x)} \, \mathrm{d}x$
   - $\int_0^1 x e^{4x^2+3} \, \mathrm{d}x$

# Homework 1

## Due: Friday, Feb 4, by the end of the class

### Instructions

- Write down the function $u$ and $v$ which you are using for substitution rule or integration by parts clearly in each problem.
- Note that for indefinite integrals, you need add constants to your final answers.

1. Integration by parts (You may also need to use substitution rule.)
    - $\int \frac{\ln x}{x^2}\, dx$
    - $\int x^2 \sin x\, dx$
    - $\int (\ln x)^2\, dx$
    - $\int \arccos x\, dx$
    - $\int e^{\sqrt{x}}\, dx$

2. Trigonometric integration: Evaluate the following integral of the form $\int \sin^n x \cos^m x\, dx$.
    - $\int \sin^2 x \cos^3 x\, dx$
    - $\int \cos^4 x\, dx$

3. Trigonometric substitution
    - $\int \frac{x^2}{\sqrt{9-x^2}}\, dx$
    - $\int \frac{1}{\sqrt{25+x^2}}\, dx$
    - $\int \frac{1}{\sqrt{x^2+2x}}\, dx$
    - $\int (x-2)^3 \sqrt{5+4x-x^2}\, dx$

# Homework 2

## Due: Friday, Feb 11, by the end of the class

Note that for indefinite integrals, you need add constants to your final answers.

1. Partial Fractions
   - $\displaystyle\int \frac{2x+5}{x^2+4x+8}\,\mathrm{d}x$
   - $\displaystyle\int \frac{2x^2-x+4}{(x^2+4)(x-1)}\,\mathrm{d}x$
   - $\displaystyle\int \frac{x}{x^4+2x^2+2}\,\mathrm{d}x$
   - $\displaystyle\int \ln(x^2+1)\,\mathrm{d}x$
   - $\displaystyle\int \frac{1}{\sqrt{x}+x\sqrt{x}}\,\mathrm{d}x$
   - $\displaystyle\int \frac{1}{x+\sqrt[3]{x}}\,\mathrm{d}x$

2. Approximate Integration
   - Use the Midpoint Rule with $n=5$ to approximate $\displaystyle\int_0^{10} x^2\,\mathrm{d}x$.
   - Use the Trapezoidal Rule with $n=6$ to approximate $\displaystyle\int_0^{\pi} \sin^2 x\,\mathrm{d}x$.

3. Improper Integrals: compute the following integrals or show that it diverges.
   - $\displaystyle\int_1^{\infty} \frac{1}{\sqrt{x}}\,\mathrm{d}x$
   - $\displaystyle\int_1^{\infty} \frac{1}{1+x^2}\,\mathrm{d}x$
   - $\displaystyle\int_{\pi}^{\infty} \sin x\,\mathrm{d}x$
   - $\displaystyle\int_e^{\infty} \frac{1}{x\ln x}\,\mathrm{d}x$
   - $\displaystyle\int_{-\infty}^{\infty} xe^{-x^2}\,\mathrm{d}x$

# Homework 3

## Due: Friday, Feb 25, by the end of the class

**Instructions**

Step 1. Write down what $\mathrm{d}s$ is before setting up an integral

Step 2. Substitute $\mathrm{d}s$ and simplify the integral as much as you can

Step 3. Compute the integral (substitution rule, integration by parts, etc.)

1. Arclength: for the following curves write down (do not evaluate) an integral w.r.t. $x$ representing the length. Then write down an integral w.r.t. $y$.
   - $y = x^3$ for $x \in [1, 2]$.
   - $y = e^x$ for $x \in [0, 2]$.
2. Arclength: compute determine the arclength of the following curves
   - $y = \dfrac{x^3}{6} + \dfrac{1}{2x} =$ for $x \in [1, 3]$.
   - $y = \cosh x$ for $x \in [0, \ln 2]$.

     The hyperbolic cosine function $\cosh x$ is given by $\cosh x = \dfrac{e^x + e^{-x}}{2}$.
   - $y = \ln(\cos x)$ for $x \in \left[0, \dfrac{\pi}{3}\right]$.
3. Area of a Surface of Revolution: determine the area of the surface obtained by rotating the curve
   - $y = \sqrt{9 - x^2}$ for $x \in [-2, 2]$, rotating about the $x$-axis.
   - $y = x^2$ for $x \in [1, 2]$, rotating about the $y$-axis.
   - $y = \dfrac{(x^2 + 2)^{3/2}}{3}$ for $x \in [1, 2]$, rotating about the $y$-axis.

     Hint: For this question, it'd be easier if you treat $\mathrm{d}s$ as $\sqrt{1 + (y')^2}\,\mathrm{d}x$.

# Homework 4

## Due: Friday, Mar 4, by the end of the class

1. Determine whether the sequence converges or diverges. If it converges, find the limit.

   You might want to review Chapter 2 in the book for computing limits.

   - $a_n = \dfrac{3+5n^2}{n+n^2}$
   - $a_n = \dfrac{2n^4-11n+5}{4n-1}$
   - $a_n = \dfrac{n^2-2n-1}{n^3+3}$
   - $a_n = \left(1+\dfrac{2}{n}\right)^n$

2. Computing Series

   - $\displaystyle\sum_{n=0}^{\infty} 9^{-\frac{n}{2}} 2^{1+n}$
   - $\displaystyle\sum_{n=5}^{\infty} \frac{3}{n^2-7n+12}$ Hint: recall $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

3. The Divergence Test: prove the following series diverges.

   - $\displaystyle\sum_{n=2}^{\infty} \cos\left(\frac{n\pi}{2}\right)$
   - $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4+e^{-n}}$
   - $\displaystyle\sum_{n=0}^{\infty} \frac{e^n}{n^3+n}$

4. The Integral Test: determine if the following series converges or diverges.

   - $\displaystyle\sum_{n=1}^{\infty} \frac{n^4}{e^n}$
   - $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^3+1}$
   - $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ for $p > 1$ and for $p \leq 1$ respectively.

## Homework 5

### Due: Friday, Apr 1, by the end of the class

1. Comparison Test for Sequence

   This question asks you to use the comparison test for a pedagogical reason. There are other ways to solve Q1, for example, using the limit comparison test.

   - $\displaystyle\sum_{n=1}^{\infty} \frac{3n-2}{2n^3+5}$

   - $\displaystyle\sum_{n=1}^{\infty} \frac{1}{e^n+n^2}$

2. The Limit Comparison Test

   - $\displaystyle\sum_{n=1}^{\infty} \frac{n^2-n+5}{n^3-3n+6}$

   - $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n+1}}$

3. Alternating Series: determine if the following series converges absolutely, conditionally or diverges.

   For this question, you may use whichever test you like. If a series converges, specify if it converges absolutely, conditionally, or both.

   - $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n n^2}{n!}$

   - $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^{2n}}{n^n}$

   - $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{2/3}}$

   - $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(\ln n)^2}{n}$

# Homework 6

## Due: Friday, Apr 8, by the end of the class

**Warning**

1. When apply convergence tests such as alternating series test for the boundary cases, make sure you have check **all** the assumptions before apply the test.
2. Simplify your final answer until it is a **single** sum of the form $\sum c_n(x-a)^n$.
3. Don't forget the range of $x$.
4. Remember to compute the constant $C$ for the term-by-term integration.

1. Power Series: determine the radius of convergence $R$ and interval of convergence $I$.
   - $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{(x-3)^n}{n \cdot 5^n}$
   - $\displaystyle\sum_{n=1}^{\infty} \frac{(x+2)^n}{2^n \ln n}$
   - $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 x^{2n}}{(2n)!}$

2. Functions as Power Series: find the power series representations using substitution, term-by-term integration and differentiation.
   - $\arctan x$ Hint: $\arctan x = \displaystyle\int \frac{1}{1+x^2}\,\mathrm{d}x$.
   - $\ln\left(\dfrac{1+x}{1-x}\right)$
   - $\dfrac{7x-x}{3x^2+2x-1}$ Hint: Decompose the rational function first.

3. Taylor and Maclaurin Series: find the power series representation for by computing the $n$-th derivative $f^{(n)}(0)$.
   - $f(x) = \sin(x)$ centered at $\dfrac{\pi}{2}$.
   - $f(x) = (1+x)^k$ centered at 0. Note that tshis proves the binomial theorem.

4. Applications:
   - Using the Maclaurin series for $f(x) = e^x$ and the alternating series estimation theorem in 11.5 to approximate $\displaystyle\int_0^1 e^{-x^2}\,\mathrm{d}x$ with error $R < 0.04$.

# Homework 7

## Due: Friday, Apr 22, by the end of the class

- Unless otherwise stated, $t \in (-\infty, \infty)$
- Q1 aims for helping you getting intuition to parameterizations. You may describe the curve by words if you find it difficult to draw.

1. Curves Defined by Parametric Equations: Draw or describe the following curves on the $x, y$-plane.
   - $x = 2t + 3, y = 3t - 4$
   - $x = t^2 - 3, y = 2t - 1$
   - $x = 3 - \cos t, y = \sin t + 1$ for $0 \leq t \leq 2\pi$
   - $x = \sqrt{t+1}, y = \dfrac{1}{t+1}$

2. Derivatives: compute $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ for the following parameterizations.
   - $x = 2t + 1, y = t^3 - 3t + 4$
   - $x = 3\cos t, y = 3\sin t + 1$ for $0 \leq t \leq 2\pi$
   - $x = te^{-t}, y = 2t^2 + 1$

3. Second order derivatives: compute $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ for the following parameterizations.
   - $x = t^2, y = t^3$
   - $x = 3t^2, y = t^4 - 8t^2$
   - $x = e^{-t}, y = t^3 + t + 1$

4. Arc Length with Parametric Curves: Compute the length of the curve
   - $x = 2\cos^2 t, y = 2\cos t \sin t$ for $0 \leq t \leq \pi$
   - $x = t - \sin t, y = 1 - \cos t$ for $0 \leq t \leq 2\pi$

# Homework 8

## Due: Friday, Apr 29, by the end of the class

1. Surface area with Parametric Curves: Compute the length of the curve
   - $x = t^2 + 3, y = 2t$ for $0 \leq t \leq 1$ rotating about $x$-axis.
   - $x = e^t - t, y = 4e^{\frac{t}{2}}$ for $0 \leq t \leq 1$ rotating about $y$-axis.

2. Polar curves: Convert the following curves to Cartesian coordinates. Then describe the curve.
   - $r = 5\cos\theta$
   - $r = 5\sec\theta$
   - $\theta = \frac{\pi}{3}$
   - $r^2\cos(2\theta) = 1$

3. Derivatives: compute $\frac{\mathrm{d}y}{\mathrm{d}x}$ for the following polar curves.
   - $r = \cos(2\theta)$
   - $r = \theta\sin(\theta)$

There will not be polar coordinates questions on the final, so I'm only asking you to set up the integral here. Note that the first three integrals is solvable using knowledge from Chapter 7.

4. Set up (no need to evaluate) the area enclosed.
   - $r = 4 + 3\sin(\theta)$, in the first quadrant
   - $r = 2 + \cos(\theta)$, in the upper half plane

5. Set up (no need to evaluate) the arc length for the following polar curves.
   - $r = 6\sin(\theta)$
   - $r = 1 + \sin\theta$