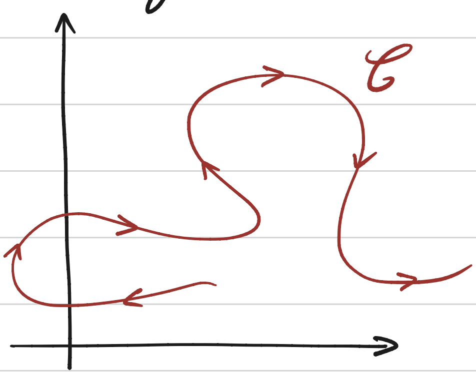
This time: curves defined by parametric eqns.
Consider a particle moves along a curve like the following. It is not possible to write $\mathcal{C}$ as

$$y = f(x)$$



because $\mathcal{C}$ fails the vertical line test. However, if we introduce a new variable t then we can write x and y-coordinates as functions which depend on time.

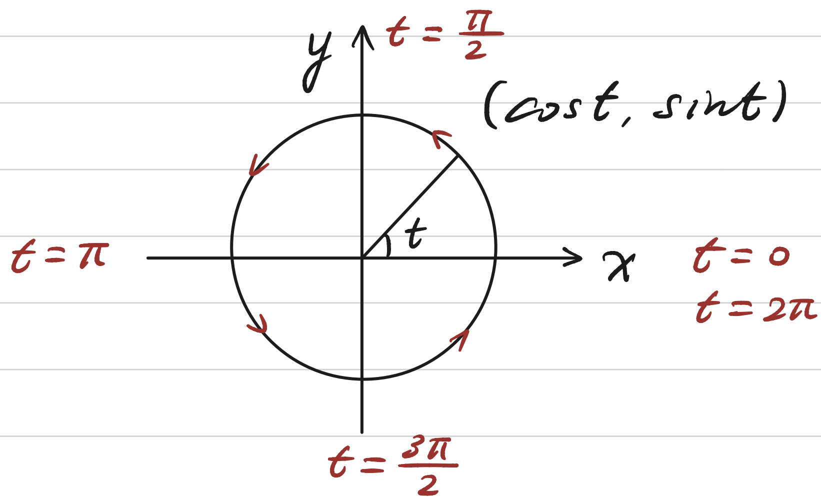
__Def__ We call t parameter

$$\left.\begin{aligned} x &= f(t) \\ y &= g(t) \end{aligned}\right\}$$ parametric equations / parametrization

$\mathcal{C}$ a parametric curve.

Example 1. Consider $x = \cos t \quad 0 \leq t \leq 2\pi$
$y = \sin t$

Note that $\cos^2 t + \sin^2 t = 1$ implies $x^2 + y^2 = 1$
So these gives us a unit circle.



Note that parameterization is not unique

Example 2. $x = \sin 2t \qquad 0 \le t \le \pi$

$y = \cos 2t$

These give the same circle but with opposite orientation.

Example 3. Find parametrization for the circle of radius r, centered at (a, b).

$$(x-a)^2 + (y-b)^2 = r^2$$

$$r\cos t \qquad r\sin t$$

scaling

$$\Rightarrow \quad x = a + r\cos t \qquad 0 \le t \le 2\pi.$$

$$y = b + r\sin t$$

translation

Example 4. $x = y^4 - 3y^2$

Take $y = t$ then $x = t^4 - 3t^2$, $t \in \mathbb{R}$

calculus with parametric curves

We can apply calculus techniques to parametrized curves.

Tangent: $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \text{if} \quad \frac{dx}{dt} \neq 0$$

The above is derived from chain rule

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

Remark:

1. $\frac{dx}{dt} \neq 0$ so that we can take quotient.

2. $\left.\begin{array}{l} \frac{dx}{dt} = 0 \\ \frac{dy}{dt} \neq 0 \end{array}\right\}$ corresponds to verticle line $y = ct$.

3. $\left.\begin{array}{l} \frac{dx}{dt} \neq 0 \\ \frac{dy}{dt} = 0 \end{array}\right\}$ corresponds to horizontal line $x = ct$.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

Arc length

$$L = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_{t_1}^{t_2} \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2}\, \underbrace{\frac{dx}{dt}\, dt}_{\text{substitution}}$$

$$= \int_{t_1}^{t_2} \underbrace{\boxed{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}}_{= ds}$$

Surface area $\quad S = \int_{t_1}^{t_2} 2\pi R\, ds$

Example $\quad \begin{cases} x = \cos^2 t \\ y = \sin^2 t \end{cases} \qquad 0 \le t \le \frac{\pi}{4}$

$$ds = \left((2\cos t(-\sin t))^2 + (2\sin t\cos t)^2\right)^{\frac{1}{2}} dt$$

$$= \left(4\cos^2 t\sin^2 t + 4\cos^2 t\sin^2 t\right)^{\frac{1}{2}} dt$$

$$= 2\sqrt{2}\cos t\sin t\, dt$$

$$= \sqrt{2}\sin(2t)\, dt$$

$$L = \int ds = \int_0^{\frac{\pi}{4}} \sqrt{2} \sin(2t)\, dt$$

$$= -\frac{\sqrt{2}}{2} \cos(2t) \Big|_{t=0}^{\frac{\pi}{4}}$$

$$= -0 - \left(-\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}$$

$$S = \int_0^{\frac{\pi}{4}} 2\pi \sin^2 t \sqrt{2} \sin(2t)\, dt$$

$$= \sqrt{2}\pi \int_0^{\frac{\pi}{4}} (1 - \cos(2t)) \sin(2t)\, dt$$

$$= \sqrt{2}\pi \left(\int_0^{\frac{\pi}{4}} \sin(2t)\, dt - \int_0^{\frac{\pi}{4}} \frac{1}{2} \sin(4t)\, dt \right)$$

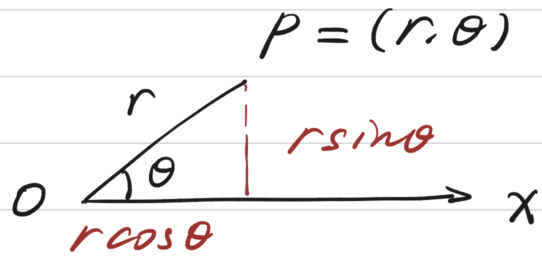
$$= \sqrt{2}\pi \left(-\frac{\cos(2t)}{2} \Big|_0^{\frac{\pi}{4}} - \frac{\cos(4t)}{8} \Big|_0^{\frac{\pi}{4}} \right)$$

$$= \sqrt{2}\pi \left(\frac{1}{2} - \left(\frac{1}{8} + \frac{1}{8}\right) \right)$$

$$= \sqrt{2}\pi \left(\frac{1}{2} - \frac{1}{4} \right) = \frac{\sqrt{2}\pi}{4}$$

Polar coordinates.

In this section we will consider a different way of writing points on the Euclidean plane.



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad\qquad \begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \arctan \frac{y}{x} \end{cases}$$

Polar $\Rightarrow$ Rectangular $\qquad$ Rectangular $\Rightarrow$ Polar

$(1,1)$ $(1,2)$

Example 1. Convert $(1,1)$ and $(1,2)$ into (r,θ) for $r>0$.

$$\left(\sqrt{2}, \frac{\pi}{4}\right) \text{ and } \left(\sqrt{5}, \arctan 2\right)$$

Polar curves

Some curves like circles or rays can be written as a simple function in terms of polar coords.

$$F(r,\theta) = 0$$

We will see how to compute arc length and surface area using polar coords.

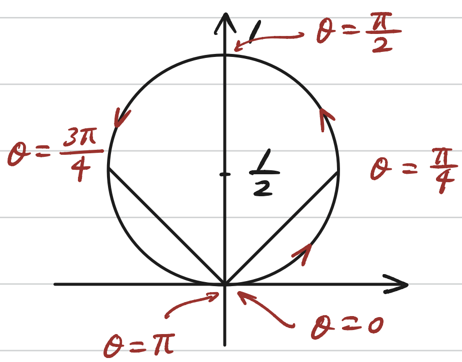
Example 2 circles

- $x^2+y^2=R^2 \iff r=R$
- $r=\sin\theta \iff r^2 = r\sin\theta$

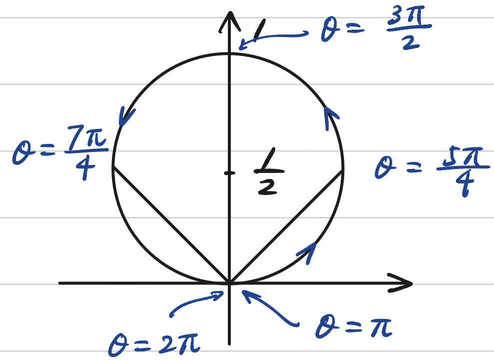
$\iff x^2+y^2=y \iff x^2+\left(y-\frac{1}{2}\right)^2=\frac{1}{4}$

Note that when $\theta\in(0,\pi)$ $r>0$

$\theta\in(\pi,2\pi)$ $r<0$

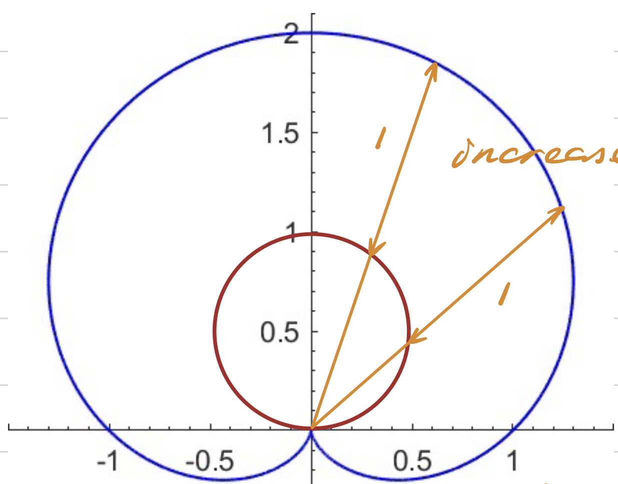


winding around the circle once



winding around the circle once again

Example 3. $r=1+\sin\theta$ cardoid



increase radius by 1.

here $\sin\theta\in(-1,0)$

$1+\sin\theta\in(0,1)$

Tangent

Now consider polar curve of the form $r = f(\theta)$ then

$$x = f(\theta)\cos\theta$$
$$y = f(\theta)\sin\theta$$

we can compute its tangent by chain rule

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

Example 4. Let $r = 1 + \sin\theta$, compute $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{\cos\theta\sin\theta + (1+\sin\theta)\cos\theta}{\cos\theta\cos\theta - (1+\sin\theta)\sin\theta}$$

$$= \frac{\cos\theta + 2\cos\theta\sin\theta}{\cos^2\theta - \sin^2\theta - \sin\theta}$$

$$= \frac{\cos\theta + \sin 2\theta}{\cos 2\theta - \sin\theta}$$

" $\frac{-1}{0}$ "

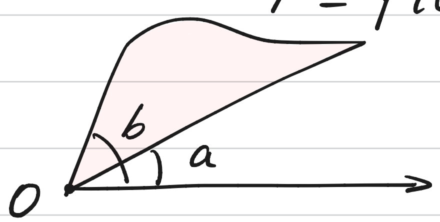
Note that $\lim_{\theta \to \left(\frac{3\pi}{2}\right)^-} \frac{dy}{dx} \overset{L'H.}{=} \lim_{\theta \to \left(\frac{3\pi}{2}\right)^-} \frac{-\sin\theta + 2\cos 2\theta}{-2\sin 2\theta - \cos\theta}$

$= -\infty$

$\sin 3\pi = 0$ $\quad \cos 3\pi = -1$

$\sin \frac{3\pi}{2} = -1$ $\quad \cos \frac{3\pi}{2} = 0$

Area



Area of sector

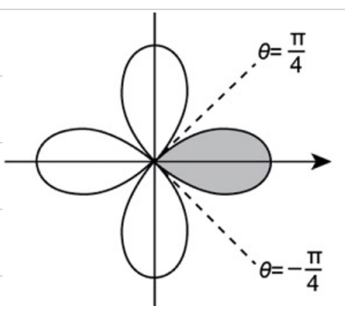
$$\Delta A \approx \frac{1}{2} r \cdot r\theta$$

Riemann sum $\quad A \approx \sum_{i=1}^{n} \frac{1}{2}\left[f(\theta_i^*)\right]^2 \Delta\theta$

taking limit gives

$$A = \int_a^b \frac{1}{2}(f(\theta))^2 \, d\theta = \int_a^b \frac{1}{2} r^2 \, d\theta$$

Example 1. Area enclosed by one loop of the four-leaved rose $r = \cos(2\theta)$.



$$A = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2} r^2 \, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \underbrace{\cos^2 2\theta}_{= \frac{1+\cos(4\theta)}{2}} \, d\theta$$

$$= \frac{1}{2}\left(\theta + \frac{\sin 4\theta}{4}\right)\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \frac{\pi}{8}.$$

Arc length

$$L = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_{\theta_1}^{\theta_2} \sqrt{\underbrace{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}_{\parallel}}\, d\theta$$

$$(r')^2 \cos^2\theta - 2rr'\cos\theta\sin\theta + r^2\sin^2\theta$$
$$+ (r')^2\sin^2\theta + 2rr'\sin\theta\cos\theta + r^2\cos^2\theta$$

$$= (r')^2 + r^2$$

$$\Rightarrow \quad L = \int_{\theta_1}^{\theta_2} \sqrt{(f(\theta))^2 + (f'(\theta))^2}\, d\theta.$$

Example 2. $r = \theta \quad 0 < \theta < 1$

$$L = \int_0^1 \sqrt{\theta^2 + 1}\, d\theta$$ trig substitution

$\theta = \tan x \qquad d\theta = \sec^2 x\, dx$

$\theta = 0 \qquad\qquad x = 0$

$\theta = 1 \quad \Rightarrow \quad x = \frac{\pi}{4}$

So $L = \int_0^{\frac{\pi}{4}} \sec x \; d(\tan x)$

To solve $I = \int \sec x \, d(\tan x)$

$= \int \sec^3 x \, dx$

Integration by parts with $u = \sec x$

$v = \tan x$

$$I = \sec x \tan x - \int \tan x \, d(\sec x)$$

$$\int \tan x \, d(\sec x) = \int \tan^2 x \sec x \, dx = \int \sec x (\sec^2 x - 1) \, dx$$

$$= \sec x \tan x - I + \underbrace{\int \sec x \, dx}_{= \ln|\sec x + \tan x| + C}$$

$$\Rightarrow \quad 2I = \sec x \tan x + \ln|\sec x + \tan x| + C$$

$$L = \int_0^{\frac{\pi}{4}} \sec x \, d(\tan x)$$

$$= \frac{1}{2}\left(\sec x \tan x + \ln|\sec x + \tan x|\right)\Big|_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\left(\sqrt{2} + \ln(1+\sqrt{2})\right)$$