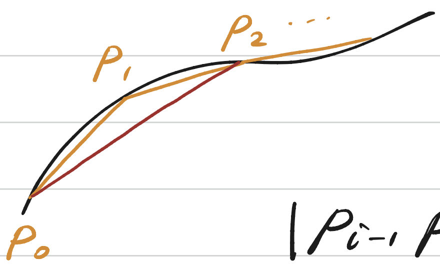
Last time: improper integral
This time: arc length

In this chapter we are going to study applications of integrals.

Lets first consider how to compute arc length



$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1} P_i|$$

$$|P_{i-1}P_i| = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$= \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\, \Delta x$$

taking limits gives f'

Let $y = f(x)$ be the curve
If f' is continuous, arc length for $x \in [a, b]$

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

Similarly, let $x = g(y)$ be the curve, then arc length for $y \in [c, d]$

$$L = \int_c^d \sqrt{1 + (g'(y))^2}\, dy$$

Example 1. $y = e^x \qquad 0 \leq x \leq 2$

$$\Rightarrow \quad L = \int_0^2 \sqrt{1+(e^x)^2}\, dx$$

or $$L = \int_1^{e^2} \sqrt{1+\left(\frac{1}{y}\right)^2}\, dy$$ as $x = \ln y$

Example 2 $\quad x^2+y^2=1$ unit circle

Using symmetry take the upper half circle

$$y = \sqrt{1-x^2} \qquad -1 \leq x \leq 1$$

$$\Rightarrow \quad L = \int_{-1}^{1} \sqrt{1+\left(\frac{2x}{2\sqrt{1-x^2}}\right)^2}\, dx$$

Arc length function

$$s(x) = \int_a^x \sqrt{1+(f'(t))^2}\, dt$$

Example 3. $f(x) = x^2 - \frac{\ln x}{8}$

$$s(x) = \int_1^x \sqrt{1 + \left(2t - \frac{1}{8t}\right)^2}\, dx$$

$$= \int_1^x \sqrt{1 + 4t^2 - \frac{1}{2} + \frac{1}{64t^2}}\, dx$$

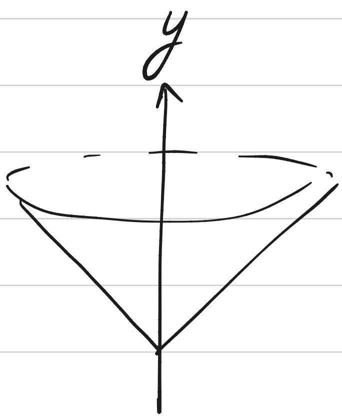
$$= \int_1^x 2t + \frac{1}{8t}\, dt$$

$$= t^2 + \ln t \Big]_1^x$$

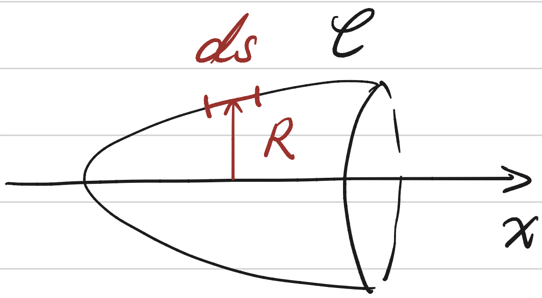
$$= x^2 + \frac{\ln x}{8} - 1$$

Surface of revolution

A surface of revolution is formed by rotating a curve about a line (usually the x or y-axis).



e.g. rotating $y=x$ about y-axis



e.g. rotating $y=\ln x$ about x-axis

To define the area, recall the surface area of a cylinder is $2\pi R l$.

If we take infinitesimal line segement ds, the small piece is approximate cylinder

$$S \approx \sum_{i=0}^{n} 2\pi f(x) \cdot ds$$

$$\Rightarrow S = \int_a^b 2\pi f(x)\, ds$$

Using arc length

$$ds = \sqrt{1+(f'(x))^2}\,dx$$

$$\Rightarrow S = \int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx$$

Given curve $y=f(x)$, rotation about x-axis

$$S = \int_a^b 2\pi y\sqrt{1+(y')^2}\,dx$$

Similarly, given curve $x=g(x)$, rotation about y-axis.

$$S = \int_c^d 2\pi x\sqrt{1+(x')^2}\,dy$$

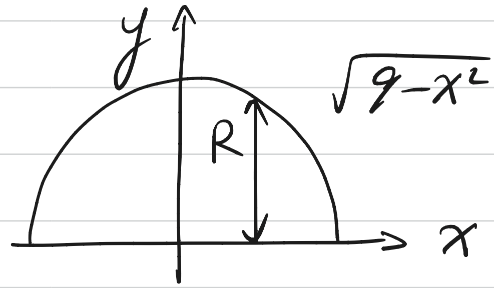
Example 1. $y = \sqrt{9-x^2}$ $\quad -2 \leq x \leq 2$
rotate about the x-axis

$$ds = \sqrt{1+(y')^2}\,dx = \sqrt{1+\left(\frac{2x}{2\sqrt{9-x^2}}\right)^2}\,dx$$

$$= \sqrt{1+\frac{x^2}{9-x^2}}\,dx = \sqrt{\frac{9-x^2+x^2}{9-x^2}}\,dx$$

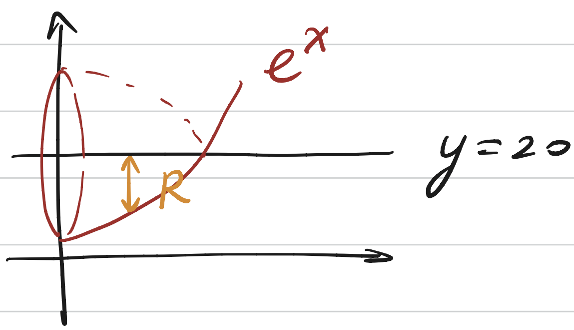
$$= \frac{3}{\sqrt{9-x^2}}\,dx$$

$$
\begin{aligned}
S &= \int_{-2}^{2} 2\pi y \, ds \\
&= \int_{-2}^{2} 2\pi \sqrt{9-x^2} \, \frac{3}{\sqrt{9-x^2}} \, dx \\
&= \int_{-2}^{2} 6\pi \, dx \\
&= 24\pi
\end{aligned}
$$



Example 2. $y = e^x \qquad 0 \leq x \leq 2$

rotate about $y = 20$



$$R(x) = 20 - e^x$$

$$
\begin{aligned}
S &= \int_{0}^{2} 2\pi R(x) \, ds \\
&= \int_{0}^{2} 2\pi (20 - e^x) \sqrt{1 - e^{2x}} \, dx
\end{aligned}
$$

Application

Example 1. hydrostatic pressure and force

$$F = mg = \rho g \overbrace{A d}^{}$$

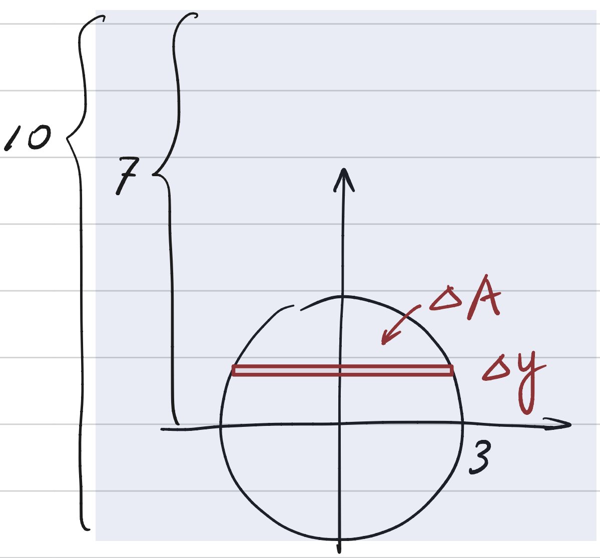
← area & width of a plate

density   acceleration due to gravity

$$P = F/A = \rho g d$$

pressure

Compute the force on one end of a cylinder with radius 3 if it is submerged in water of depth 10



$$F = \rho g A d \qquad d = 7 - y$$

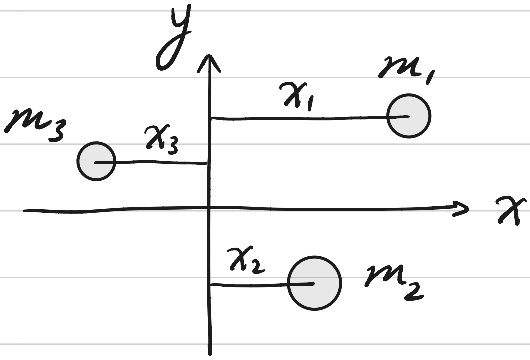
$$\Delta A \approx 2\sqrt{9 - y_i^2}\, \Delta y$$

$$dA = 2\sqrt{9 - y^2}\, dy$$

$\Delta y \to 0$

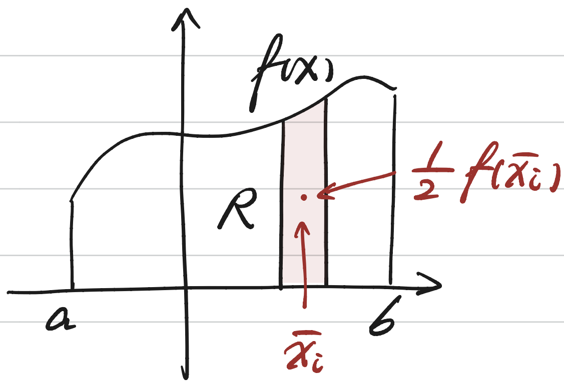
$$\Rightarrow \quad F = \int_{-3}^{3} (7-y)\, \rho g \sqrt{9-y^2}\, dy$$

Example 2 moments and center of mass.



(moment) $M_y = \sum_i m_i x_i$

(center) $\bar{x} = \dfrac{\sum_i m_i x_i}{\sum_i m_i} = \dfrac{M_y}{m}$



$$m = \rho A = \rho \int_a^b f(x)\,dx$$

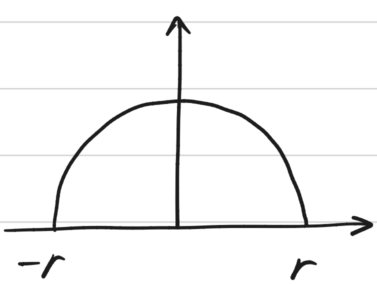
$$M_y = \rho \int_a^b x\, f(x)\,dx$$

$$M_x = \rho \int_a^b \frac{1}{2}(f(x))^2\,dx$$

$$\bar{x} = \frac{M_y}{m} \qquad \bar{y} = \frac{M_x}{m}$$

here $y_i = \frac{1}{2} f(x)$ since we are taking the center of rectangle

Find the center of mass of a semicircular plate



$$\bar{y} = \frac{1}{A}\int_{-r}^{r} \frac{1}{2}(f(x))^2\,dx$$

$$= \frac{1}{\frac{1}{2}\pi r^2}\,\frac{1}{2}\int_{-r}^{r}(r^2 - x^2)\,dx$$

$$= \frac{2}{\pi r^2}\int_0^r r^2 - x^2\,dx$$

$$= \frac{2}{\pi r^2}\left[r^2 x - \frac{x^3}{3}\right]_0^r = \frac{4r}{3\pi}$$

Summary

Infinitesimal line element

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

Arc length

$$L = \int ds$$

Infinitesimal area element

$$dA = 2\pi R\, ds.$$

distance between point on curve and rotating axis

Surface area (by revolution)

$$S = \int dA.$$

Two particular cases

about x-axis $\quad S = \int 2\pi y\, ds$

about y-axis $\quad S = \int 2\pi x\, ds$