Contents

Instructions

Deadline & Late Submissions

1. The deadline is strict. Submit your work on time to avoid penalties.
2. Late submissions are allowed with a **two-day grace period**, always receive **50% of the grade earned**, regardless of the reason.

   For example, if an assignment is due on Monday at 10:50 AM, it can be submitted by Wednesday at 10:50 AM, but will receive only 50% of the grade earned.
3. **No points** will be awarded after the solution is posted.

Submission Format

1. You are required to submit a **hard copy, not an electronic file**.
2. You have one week to complete and print your homework. Make sure you print it before the deadline, as late submissions will incur a penalty.

Extensions

1. If you need an extension due to medical reasons, you must provide a **DRES letter** or **official medical documentation before the deadline**.
2. You may submit the homework before the solution is posted.
3. If no documentation is provided, the initial instructions for late submission penalties will apply.

Extra Practice Problems

1. Questions marked with a $\star$ are extra practice problems.
2. They will **not be graded**, so there is **no need to submit** them.
3. These problems may appear on the exam.
4. Solutions will be posted.

Grading

1. If a question specifies a method to use, only answers following that method will be eligible for full credit.
2. To receive full marks, provide a **rigorous justification** for all answers:
   - Clearly **state the name of any theorems or tests** you apply.
   - **Verify all conditions** of the theorem before using it.

Homework 0 — Calculus I Review— No need to hand in

This assignment is intended to check your understanding of the topics in Calculus I. It will not be graded, and there is no need to submit it.

1. Calculating Limits
   (a) $\lim_{x\to 2} \frac{x^2+x-6}{x-2}$
   (b) $\lim_{h\to 0} \frac{(x+h)^2-x^2}{h}$
   (c) $\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{x^2+x}\right)$
2. The Chain Rule
   (a) $\frac{\mathrm{d}}{\mathrm{d}x} \ln(x+\sin x)$
   (b) $\frac{\mathrm{d}}{\mathrm{d}x} \cos(x^2 e^x)$
3. Implicit Differentiation: Solve for $\frac{\mathrm{d}y}{\mathrm{d}x}$ for the following implicit function.
   (a) $x^2+y^2=r^2$, where r is a constant
   (b) $\frac{x+y}{x-y} = x$
4. Linear Approximations and Differentials: Find the Taylor polynomials of degree two approximating the given function centered at the given point.
   (a) $f(x) = \sin(2x)$ at $a = \frac{\pi}{2}$
   (b) $f(x) = e^x$ at $a = 1$
5. Mean Value Theorem: Determine if the Mean Value Theorem can be applied to the following function on the the given closed interval.
   (a) $f(x) = 3+\sqrt{x}, x \in [0,4]$
   (b) $f(x) = \frac{x}{1+x}, x \in [1,3]$
6. L'Hospital's Rule
   (a) $\lim_{x\to 2} \frac{x^3-7x^2+10x}{x^2+x-6}$
   (b) $\lim_{x\to\infty} (e^x+x)^{\frac{1}{x}}$
   (c) $\lim_{x\to\infty} x \ln\left(1+\frac{3}{x}\right)$
7. The Fundamental Theorem of Calculus: Find the derivative of the following
   (a) $\int_1^x \frac{1}{t^3+1}\,\mathrm{d}t$
   (b) $\int_1^{\sqrt{x}} \sin t\,\mathrm{d}t$
   (c) $\int_x^{2x} t^3\,\mathrm{d}t$

8. Substitution Rule

   (a) $\displaystyle\int_{\frac{1}{2}}^{0} \frac{x}{\sqrt{1-4x^2}}\,\mathrm{d}x$

   (b) $\displaystyle\int_{\frac{1}{4}}^{\frac{1}{2}} \frac{\cos(\pi x)}{\sin^2(\pi x)}\,\mathrm{d}x$

   (c) $\displaystyle\int_{0}^{1} xe^{4x^2+3}\,\mathrm{d}x$

Homework 1 — §7.1–§7.2 — Due Jan 31, Fri 10:50am

1. Integration by parts: Evaluate the following integrals. The substitution rule may be required for some problems.

   > Write down the functions u and v used for the substitution rule or integration by parts in each problem.

   (a) (3 points) $\displaystyle\int \frac{\ln x}{x^2}\,\mathrm{d}x$

   (b) (3 points) $\displaystyle\int x^2 \sin x\,\mathrm{d}x$

   (c) (3 points) $\displaystyle\int \arccos x\,\mathrm{d}x$

   (d) (3 points) $\displaystyle\int e^{\sqrt{x}}\,\mathrm{d}x$

2. In this question, we use integration by parts to compute $I = \displaystyle\int \sin^2 x\,\mathrm{d}x$.

   (a) (3 points) i. **Step 1**. Apply the integration by parts formula to rewrite I as an expression involving $\displaystyle\int \cos^2 x\,\mathrm{d}x$.

   ii. **Step 2**. Evaluate I using the result from Step 1 and the trig identity.

   (b) (3 points) i. **Step 1**. Now apply the half-angle formula $\sin^2 x = \dfrac{1-\cos(2x)}{2}$ and use your knowledge from Calculus I to solve I again.

   ii. **Step 2**. Check that the two methods give you the same answer.

3. Trigonometric integration: Evaluate the following integrals of the form $\displaystyle\int \sin^n x \cos^m x\,\mathrm{d}x$.

   (a) (3 points) $\displaystyle\int \sin^2 x \cos^3 x\,\mathrm{d}x$

   (b) (3 points) $\displaystyle\int \cos^4 x\,\mathrm{d}x$

4. In this question, we use trig integrals to compute $I = \tan^3 x\,\mathrm{d}x$

   (a) (3 points) Evaluate $\int \tan x\,\mathrm{d}x$ using the substitution rule.

   (b) (3 points) Write $\tan^2 x$ in terms of $\sec x$. Then evaluate the integral I.

5. $\star$ A more challenging trig integration: Evaluate $\displaystyle\int \frac{\cos^3 x}{\sqrt{\sin x}}\,\mathrm{d}x$.
   (Hint: $\cos x$ has an odd power).

Homework 2 — §7.3–§7.7 — Due Feb 10, Mon 10:50am

Please read the instructions carefully and show all your work to receive full credit.

- If a question instructs you to use a certain method, only answers that follow the specified method will be eligible for full credit.
- For indefinite integrals, remember to include the constant C in your final answers.

1. Trigonometric substitution

   - Trig substitution often leads to a trig integral from §7.2. You may use previously computed integrals (from lecture notes, discussion worksheets, or homework) without justification.
   - Simplify the expression during the back-substitution step.

   (a) (3 points) $\displaystyle\int \frac{x}{\sqrt{3-2x-x^2}}\,\mathrm{d}x$
   (This is Example 3.3 in the lecture notes. Read the solution and redo the problem.)

   (b) (3 points) $\displaystyle\int x^2\sqrt{4-x^2}\,\mathrm{d}x$
   (Hint: You might need $\sin(4\theta) = 2\sin(2\theta)\cos(2\theta) = 2\sin\theta\cos\theta(\cos^2\theta - \sin^2\theta)$.)

   (c) (3 points) $\displaystyle\int \frac{1}{\sqrt{4+x^2}}\,\mathrm{d}x$

   (d) (3 points) $\displaystyle\int \frac{1}{\sqrt{2x^2+4x}}\,\mathrm{d}x$

   (e) (3 points) $\displaystyle\int (x-2)^3\sqrt{5+4x-x^2}\,\mathrm{d}x$

2. Partial Fractions

   (a) (3 points) $\displaystyle\int \frac{2x^2-x+4}{(x^2+4)(x-1)}\,\mathrm{d}x$

   (b) (3 points) $\displaystyle\int \frac{x}{x^4+2x^2+2}\,\mathrm{d}x$ (Hint: Do you need to perform the decomposition step?)

3. In this question, we evaluate the integral $\displaystyle\int \ln(x^2+1)\,\mathrm{d}x$.

   (a) (1 point) Apply the integration by parts formula to reexpress the integral.

   (b) (2 points) After completing part (a), you will obtain an integral of a rational function. Use techniques from §7.4 to evaluate the remaining integral and complete the computation.

4. In this question, we evaluate the integral $\displaystyle\int \frac{1}{\sqrt{x}+x\sqrt{x}}\,\mathrm{d}x$

   (a) (2 points) Use an appropriate substitution to get rid of $\sqrt{x}$.

   (b) (1 point) After completing part (a), you will obtain an integral of a rational function. Use techniques from §7.4 to evaluate the remaining integral and complete the computation.

5. (3 points) Approximate Integration

   (a) Use the Midpoint Rule with $n = 5$ to approximate $\displaystyle\int_0^{10} x^2\,\mathrm{d}x$.

   (b) Use the Trapezoidal Rule with $n = 6$ to approximate $\displaystyle\int_0^{\pi} \sin^2 x\,\mathrm{d}x$.

Homework 3 — §7.8–§8.2 — Due Friday 10:50am

For Q1–3:

- Review limit techniques from Calculus I and the steps for improper integrals and the comparison test in lecture notes.
- Read the remaining examples in §7.8 of the lecture notes.

1. Use the **definition** of improper integrals to determine whether each given integral converges. If the integral converges, compute its exact value.
   (a) (3 points) $\int_{-\infty}^{\infty} \sin x \, \mathrm{d}x$
   (b) (3 points) $\int_{0}^{\infty} 2xe^{-x^2} \, \mathrm{d}x$

For Q2–3, you may use any method to determine whether the improper integral converges or diverges.

2. (4 points) Decide for which values of p the improper integral $\int_{e}^{\infty} \frac{1}{x(\ln x)^p} \, \mathrm{d}x$ converges.
3. (4 points) Determine whether the improper integral $\int_{0}^{1} \frac{1+\sin^2(x)}{\sqrt{x}} \, \mathrm{d}x$ converges.
4. ⋆ Decide if the improper integral $\int_{0}^{1} \frac{\sin^2(x)}{x^{3/2}} \, \mathrm{d}x$ converges.
5. Arc Length I: For the following curves, **write down, but do not evaluate**, an integral representing the arc length.
   (a) (4 points) $y = e^x$ for $x \in [0, 2]$. Write the integrals w.r.t. x **and** y, respectively.
   (b) (2 points) The hyperbola $xy = 1$ between $(1, 1)$ and $(3, \frac{1}{3})$. Write an integral w.r.t. x.
6. Arc Length II: Compute the arc length of the following curves.
   (a) (5 points) $y = \frac{x^3}{6} + \frac{1}{2x} =$ for $x \in [1, 3]$.
   (b) (5 points) $y = \sqrt{4 - x^2}, -1 \leq x \leq 1$.

Homework 4 — §8.3–§10.3 — Due Friday 10:50am

- **Questions marked with** ⋆ are extra practice problems. They won't be graded, so there's **no need to turn them in**. However, they may appear on an exam. Solutions will be posted.
- Pay attention—Q1–2 are based on concepts from Chapter 8 and should be solved **without** using polar coordinates or parametrization. In contrast, Q3–6 require parametrization.

1. Consider the curve $x = e^{2y}$, $0 \le y \le 2$, rotated about the y-axis.
   (a) (4 points) Set up and **evaluate** an integral with respect to x that represents the area of the resulting surface.
   (b) ⋆ Set up, **but do not evaluate** an integral with respect to y that represents the area of the resulting surface. (Hint: first find the inverse function.)
2. Determine the area of the surface obtained by rotating the following curve:
   (a) (5 points) $y = \sqrt{9 - x^2}$ for $x \in [-2, 2]$, rotated about the x-axis.
   (b) (5 points) $y = x^2$ for $x \in [1, 2]$, rotated about the y-axis.
   (c) ⋆ $y = \dfrac{(x^2+2)^{3/2}}{3}$ for $x \in [1, 2]$, rotated about the y-axis.
3. ⋆ We won't have time to cover §8.3 in the lecture. Please read the examples in the lecture notes on hydrostatic pressure, force and moments, and center of mass. If you have any questions, I'm happy to discuss them.
4. ⋆ For each described scenario, find a parametrization of the point's path in the xy-plane. **Don't forget to include the bounds for your parameter!**
   (a) A point starts at $(2, 0)$ and travels exactly once around the circle $x^2 + y^2 = 4$ in the counterclockwise direction. (The point returns to its starting position.)
   (b) A point starts at $(2, 0)$ and travels exactly three times around the circle $x^2 + y^2 = 4$ in the counterclockwise direction. (The point returns to its starting position.)
   (c) A point starts at $(0, 2)$ and travels exactly once around the circle $x^2 + y^2 = 4$ in the counterclockwise direction. (The point returns to its starting position.)
   (d) A point starts at $(0, 2)$ and travels exactly once around the circle $x^2 + y^2 = 4$ in the clockwise direction. (The point returns to its starting position.)
   (e) A point starts at $(0, 0)$ and travels exactly once around the circle $(x - 2)^2 + y^2 = 4$ in the counterclockwise direction. (The point returns to its starting position.)
5. (8 points) Curves Defined by Parametric Equations: Draw or describe the following curves on the xy-plane.
   (a) $x = 2t + 3, y = 3t - 4$.
   (b) $x = t^2 - 3, y = 2t - 1$.
   (c) $x = 3 - \cos t, y = \sin t + 1$ for $0 \le t \le 2\pi$.
   (d) $x = \sqrt{t+1}, y = \dfrac{1}{t+1}$.
6. (4 points) Compute the arc length of the parametric curve

$$x = e^t - t, \quad 0 \le t \le 2$$
$$y = 4e^{t/2}.$$

   (Hint: When simplifying $\mathrm{d}s$, the technique used in Q2(a) from w5-2 would be helpful.)

7. (4 points) Compute the surface area of a sphere with radius $\rho > 0$ using parametrization.
(Hint: The sphere is formed by rotating the semicircle

$$x = \rho \cos t, \quad -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$$
$$y = \rho \sin t.$$

about the y-axis.)

8. $\star$ Consider the parametric curve

$$x = \cos^3 \theta, \quad 0 \leq \theta \leq \frac{\pi}{2}$$
$$y = \sin^3 \theta.$$

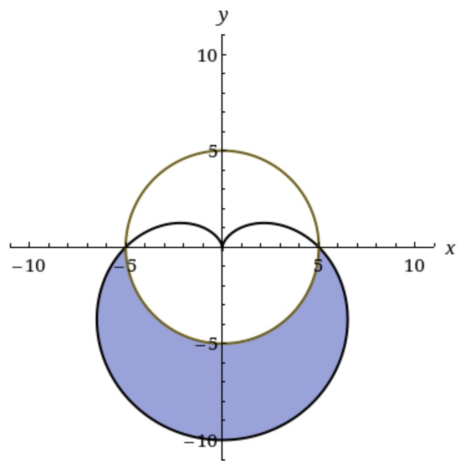
   (a) Compute the arc length of the given curve.
   (b) Rewrite the parameterization as an implicit equation involving only x and y. (Hint: Use a trigonometric identity.)

Homework 5 — §10.4–§11.2 — Due Friday 10:50am

1. (a) (3 points) Determine the area of the region enclosed by the polar curve $r = \csc\theta$, along with the lines $\theta = \frac{\pi}{3}$ and $\theta = \frac{\pi}{2}$.

   (b) (3 points) Verify the obtained result using geometric arguments.
   (Hint: Consider sketching the given region. You should recognize that it corresponds to a right triangle. If the interpretation of the curve $r = \csc\theta$ is unclear, express it in Cartesian coordinates using the definition $\csc\theta = \frac{1}{\sin\theta}$.)

2. Consider the region that lies inside the polar curve $r = 5 - 5\sin\theta$ and outside the circle $r = 5$.



   (a) (1 point) Find the range of r and θ.
   (Hint: Your answer should be of the form $a \leq r \leq b, c \leq \theta \leq d$.)

   (b) (4 points) Compute the area of the bounded region.

3. ⋆ This question introduces/reviews computations about factorials. For a non-negative integer n, we define $n!$ to be the product of all positive integers less than or equal to n, i.e.

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$$

   For example, $3! = 3 \times 2 \times 1 = 6, 4! = 4 \times 3 \times 2 \times 1 = 24$. By convention, we define $0! = 1$.

   (a) Compute 5!.

   (b) What is $\frac{n!}{(n-1)!}$? What is $\frac{(3n+1)!}{(3n-2)!}$?

4. ⋆ For each of the following sequences, perform the following tasks:

   - Derive a formula for the general term a_n, assuming that the index n starts from 1.
   - Evaluate the limit $\lim_{n\to\infty} a_n$.

   (a) For this part, first rewrite every term as a power of 3 and find the pattern of the exponential. For example, the third term here should be $3^{7/8}$.

$$\{\sqrt{3}, \sqrt{3\sqrt{3}}, \sqrt{3\sqrt{3\sqrt{3}}}, \sqrt{3\sqrt{3\sqrt{3\sqrt{3}}}}, \ldots\}$$

(b)

$$\{-3, 2, -\frac{4}{3}, \frac{8}{9}, -\frac{16}{27}, \frac{32}{81}, \ldots\}$$

- We didn't have enough time to go through Example 2.7 and Example 2.9 in class. You may find it useful to read those examples in the lecture notes before doing the following questions.
- For the following questions. You need to provide a rigorous justification for your answer.

5. For each of the following sequences, determine the limit $\lim_{n\to\infty} a_n$.
   (a) (3 points) $a_n = \dfrac{\cos^2(n)}{e^n}$
   (b) (3 points) $a_n = \dfrac{n}{3}\sin(\dfrac{1}{n})$ (Hint: $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$.)
   (c) (3 points) $a_n = \dfrac{n!}{n^n}$
6. Determine the sum of the following series.
   (a) (3 points) $\displaystyle\sum_{n=0}^{\infty} 9^{-\frac{n}{2}} 2^{1+n}$.
   (b) (3 points) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+3n+2}$. (Hint: apply partial fraction decomposition.)
7. Determine whether the following series converges. You do not need to find the sum.
   (a) (2 points) $\displaystyle\sum_{n=0}^{\infty} \cos(n\pi)$.
   (b) (2 points) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{4n+7}$.

Homework 6 — §11.3–§11.4 — Due Friday 10:50am

1. (4 points) Determine for which values of $p > 0$ the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converges.
2. Consider the series $\displaystyle\sum_{n=1}^{\infty} 2ne^{-n^2}$.
   (a) (4 points) Use the Integral Test to show this series converges.
   (b) (4 points) Find the smallest N such that $|S - S_N| \leq e^{-(231)^2}$. Recall the estimate for the Integral Test says $R_N \leq \int_N^\infty f(x)\, \mathrm{d}x$.
3. Use the direct comparison test (DCT) to decide whether the series converges or diverges.
   (a) ⋆ $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n} - 1}$.
   (b) (4 points) $\displaystyle\sum_{n=1}^{\infty} \frac{n \sin^2(n)}{n^3 + 1}$.
   (c) ⋆ $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan(n) + 5^n}{7^n}$.
   (d) (4 points) $\displaystyle\sum_{n=1}^{\infty} \frac{5^n + 2^n}{3^n - 1}$.
4. Use the limit comparison test (LCT) to decide whether the series converges or diverges.
   (a) (4 points) $\displaystyle\sum_{n=1}^{\infty} \frac{e^n + 1}{ne^n + 1}$.
   (b) ⋆ $\displaystyle\sum_{n=1}^{\infty} \sin(\frac{1}{n})$. (Hint: $\displaystyle\lim_{n\to\infty} n \sin(\frac{1}{n}) = 1$.)
   (c) (4 points) $\displaystyle\sum_{n=1}^{\infty} \frac{2n^3 + 1}{n^5 + 4n^2 + 3}$
   (d) ⋆ $\displaystyle\sum_{n=1}^{\infty} \frac{1 + 3^n}{4 + 2^n}$

Homework 7 — §11.5–§11.6 — Due Friday 10:50am

1. Consider the alternating series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{5n^2+20}$.
   (a) (3 points) Use **the Alternating Series Test** to show this series converges.
   (b) (2 points) Find the smallest integer N for which S_N is guaranteed to estimate the sum S of the series with an error of no more than $\dfrac{1}{1000}$ by the Alternating Series Estimation Theorem.

2. Use either **the Ratio Test or the Root Test** to determine whether each of the following series is absolutely convergent, conditionally convergent, or divergent.
   (a) (4 points) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \left(\frac{\ln n}{4n}\right)^n$
   (b) (4 points) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{e^{n^2}}$
   (c) (4 points) $\displaystyle\sum_{n=1}^{\infty} \frac{1\cdot 4\cdot 7\cdot 10\cdots(3n+1)}{3\cdot 5\cdot 7\cdot 9\cdots(2n+3)}$

3. In this question, we test whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(7n)}{1+3^n}$ is absolutely convergent, conditionally convergent or divergent.
   (a) (4 points) We first check whether $\displaystyle\sum_{n=1}^{\infty} \left|\frac{\sin(7n)}{1+3^n}\right|$ is convergent. Apply **the Direct Comparison Test** to show this series is convergent.
   (b) (1 point) Based on your answer in (a), determine whether the series is absolutely convergent, conditionally convergent, or divergent.

4. In this question, we test whether the series $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ is absolutely convergent, conditionally convergent, or divergent.
   (a) (4 points) Apply the **Direct Comparison Test** to show that the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$ diverges. (Hint: When does $\ln n < n$?)
   (b) (3 points) Apply the **Alternating Series Test** to determine whether $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ converges.
   (c) (1 point) Based on the previous parts, determine whether the series is absolute convergent, conditionally convergent or divergent.

5. ⋆ This question gives an example where the ratio test doesn't work.
   Consider the series $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{(c_n)^n}{n}$, where $c_n = \dfrac{1}{2} + (-1)^n \dfrac{1}{2n}$.
   (a) Convince yourself the series can be rewritten as $\displaystyle\sum_{n=1}^{\infty} a_n$, where $a_n = (-1)^n \dfrac{1}{n}\left(\dfrac{1}{2} + (-1)^n \dfrac{1}{2n}\right)^n$. We will be applying the Ratio test to $\sum a_n$ in part (c).
   (b) Compute $\displaystyle\lim_{n\to\infty}\left|\frac{a_{2n+1}}{a_{2n}}\right|$ and $\displaystyle\lim_{n\to\infty}\left|\frac{a_{2n}}{a_{2n-1}}\right|$.

(c) *Here's a way to show the limit of a sequence doesn't exist: for a sequence* b_n*, if* $\lim_{n\to\infty} b_{2n}$ *and* $\lim_{n\to\infty} b_{2n-1}$ *both exist, but they are not equal, then* $\lim_{n\to\infty} b_n$ *doesn't exist. (We have seen this in the proof of the Alternating Series test).*

Apply this result to conclude that $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ doesn't exist, so the Ratio Test doesn't work.

(d) Check that $\lim_{n\to\infty} c_n = \frac{1}{2}$, and conclude using the Divergence Test that the series diverges.

Homework 8 — §11.8–§11.9 — Due Friday 10:50am

1. For each of the following power series, determine the **radius of convergence** R and the **interval of convergence** I for the power series.
   (a) (5 points) $\sum_{n=1}^{\infty} \frac{x^{2n}}{(-4)^n}$
   (b) (5 points) $\sum_{n=2}^{\infty} \frac{(x+2)^n}{2^n \ln n}$
   (c) (4 points) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

2. Consider the power series $\sum_{n=1}^{\infty} \frac{n!}{n^n} x^n$.
   (a) (4 points) Decide the radius of the convergence of the power series.

   The following parts will give you an idea of the types of questions in a proof-based course.

   (b) ⋆ Prove the following inequality: $\int_1^n \ln x\, dx \leq \sum_{i=1}^{n} \ln(i) \leq \ln(n) + \int_1^n \ln x\, dx$. (Hint: Use left and right Riemann sums.)
   (c) ⋆ Deduce from the above part that $n^n e^{-n+1} \leq n! \leq n \cdot n^n \cdot e^{-n+1}$.
   (d) ⋆ From the left side of the inequality, we notice that $\frac{n!}{n^n} e^n \geq e$. Use this to decide the interval of convergence of the power series.

3. Find the power series representation for each of the following functions and determine their radius of convergence. You may use any results we have derived in class.
   (a) (4 points) $f(x) = \frac{1}{1+x^2}$.
   (b) (4 points) $f(x) = \arctan x$. (Hint: part (a)).
   (c) (4 points) $f(x) = \frac{2x+3}{x^2+3x+2}$. (Hint: partial fractions)

Homework 9 — §11.10–§11.11 — Due Monday 10:50am

1. ⋆ Find the Taylor series representation by computing the n-th derivative $f^{(n)}(0)$.
   (a) $f(x) = \sin(x)$ centered at $\frac{\pi}{2}$.
   (b) $f(x) = (1+x)^k$ centered at 0.
2. Consider the binomial expansion
$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n, \quad |x| < 1,$$
   where $\binom{k}{n} = \dfrac{k(k-1)\cdots(k-n+1)}{n!}$.
   (a) Use the above equation to express $f(x) = \sqrt{1+x^2}$ as a Maclaurin series. Do not simplify the binomial coefficients at this stage.
   (b) Compute and simplify the **first four** nonzero terms of the series obtained in part (a), that is, for $n = 0$ through $n = 3$.
   (c) Using the series expansion from part (b), determine the sixth derivative $f^{(6)}(0)$.
3. Evaluate each of the following series by identifying and applying a known power series expansion. (No points will be given for any other methods.)
   (a) $1 + 1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \cdots$
   (b) $\pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \frac{\pi^7}{7!} + \cdots$
   (c) $\frac{\pi}{2} - \frac{\pi^3}{2^3 \cdot 3!} + \frac{\pi^5}{2^5 \cdot 5!} - \cdots$
4. ⋆ Use the Limit Comparison Test (LCT) and the Maclaurin series to guide your choice of comparison series.
   (a) For $\displaystyle\sum_{n=1}^{\infty} \arctan\left(\frac{1}{n^2}\right)$, choose an appropriate b_n.
   (b) For $\displaystyle\sum_{n=3}^{\infty} \ln\left(1 + \frac{1}{n(\ln n)^2}\right)$, what should b_n be?
5. Use multiplication/division of series to compute the Maclaurin expansion to the specified order.
   (a) $f(x) = e^{x^2/2} \cos x$, to order 4.
   (b) $f(x) = \dfrac{x}{1 + \ln(1+x)}$, to order 3.
6. In this question, we want to approximate $\sqrt{1.1}$ using Taylor polynomials.
   (a) Find the second-degree Taylor polynomial $T_2(x)$ of $f(x) = \sqrt{x}$ at $a = 1$.
   (b) Use $T_2(x)$ to estimate $\sqrt{1.1}$.
   (c) Find the maximal value of $|f^{(3)}(x)|$ on the interval $[1, 1.1]$.
   (d) Find an upper bound of $|\sqrt{1.1} - T_2(1.1)|$ given by the Taylor's Inequality.
7. ⋆ Using the Maclaurin series for $f(x) = e^x$ and the alternating series estimation theorem to approximate $\displaystyle\int_0^1 e^{-x^2}\,dx$ with error $R < 0.04$.