Fall 2021 MATH241 Discussion

This document can be found on my website, named as "Discussion notes". If needed, I will update further information in the same document.

Time & location

- Section BDH: TR 3-4pm, 441 Altgeld.
- Section BDI: TR 4-5pm, 441 Altgeld.

Contact

- Email: xinran4@illinois.edu. Please included MATH241 and your section number in your email subject. If you don't get a reply in two days, feel free to send me a reminder.
- **Any math question should be post on Campuswire** because it can be answered much quicker than email and we TAs can type math symbols on Campuswire.
- Office hour: Wed 4-6pm on Zoom.[1]

Covid related

- **Face mask is required** all the time during discussion.
- Face shield is in general not acceptable unless one holds a DRES accommodation letter.[2]
- **According to university policy, student who does not wear a proper face covering will be asked to put on one or to leave the class room.**
- **If the student refuse to leave, I'll have to dismiss the class and report this to the undergraduate office.**
- In the case one tested positive, status can be checked by your instructor. I'll have to verify it before giving any excuse of being absent due to covid.[3]

Worksheet

- I will print copies and bring it to the classroom.
- You'll find group number written at the up right corner.
- If you prefer to work on a electronic version, you can find the worksheets on Moodle under "Worksheets" folder.
- The worksheets are long and you might not be able to finish them in calss, so you don't have to write down every single step.
- **Solutions will be avaliable on Moodle at 5pm** after discussion sections are done.
- Ask for hints when you get stuck on a problem.

[1]Office hour is run for all students in MATH241, regardless of section.
[2]See accommodation below for more information.
[3]Also see the grading section below.

Grading

- **Attendance is required** in order to get full grade for discussion.
- You shouldn't come to the class if you are sick.
- **The lowest 4 scores will be dropped** in order to remediate unforeseen illness, change of location or any possible reason for missing a class.
- If you were ill for more than 4 classes and want to see if you could be excused from that, you'll need to provide documents such as DRES letter to your instructor.
- Worksheets will be graded in a scale of 0-5. They are **not** graded for correctness.
- Most likely you'll get a full mark. In case you are interested, here is a sample grading scale:

  5 Most likely you'll get a full mark

  4 Being late or leave early for 15 min

  3 Being late or leave early for 25 min

  2 Being late or leave early for 35 min

  1 Not doing anything at all during the class

  0 Not showing up for any reason.

Accommodation

- Please contact the Disability Resources & Educational Services (DRES), if you need any sort of accommodation.
- You'll need to email **both your instructor and me** once you get the accommodation letter.

Contents

Worksheet 1: Calculus II review

Chain rule

If $h(x) = g \circ f(x)$, then

$$h'(x) = g'(f(x)) \cdot f'(x).$$

Arc length of parameterized curve

Given a parameterized curve $(x(t), y(t))$, then the arc length between $(x(a), y(a))$ and $(x(b), y(b))$ is

$$s = \int_a^b \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t.$$

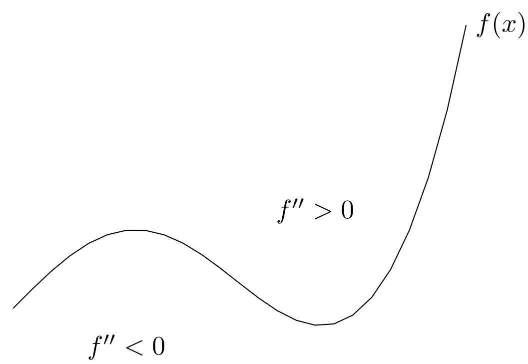
First and second derivative tests

We use the first and second derivative tests to determine local minimum and maximum.

First derivative tests. Compute $f'(x) = 0$ to find critical points.

Second derivative tests.

- If $f''(x) > 0$ for all x in the interval, then f is concave upward $\implies$ local minimum.
- If $f''(x) < 0$ for all x in the interval, then f is concave downward $\implies$ local maximum.



Taylor series

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \end{aligned}$$

Substitution rule/Change of variable

Let $u = g(x)$, then

$$\int f(g(x)) \cdot g'(x)\,\mathrm{d}x = \int f(u)\,\mathrm{d}u.$$

Open discussion: Sep 2nd

1. Find the angle between the planes $2x + y + 2z = 3$ and $4y + 3z = 1$.

   Hint: Use normal vectors.

2. Determine if the following vectors lie in the same plane or not:

   $$\mathbf{a} = \langle 1, 4, -7 \rangle, \ \mathbf{b} = \langle 2, -1, 4 \rangle \ \text{ and } \ \mathbf{c} = \langle 0, 9, -18 \rangle.$$

   Hint: Consider $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$, draw a picture represent $\mathbf{b} \times \mathbf{c}$.

3. Find a counter example for the following:

   $$\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c} \implies \mathbf{b} = \mathbf{c}.$$

   Hint: Consider a triangle ABC. Let $\overrightarrow{AB} = \mathbf{a}$, $\overrightarrow{CA} = \mathbf{b}$ and $\overrightarrow{CB} = \mathbf{c}$. What is $\mathbf{a} \times (\mathbf{b} - \mathbf{c})$?

4. Derive the following formula:

   - Area of a parallelogram determined by the vectors $\mathbf{a}$ and $\mathbf{b}$:

     $$\text{Area} = |\mathbf{a} \times \mathbf{b}|.$$

     Hint: Notes that $\mathbf{a} \times \mathbf{b} = |\mathbf{a}| \cdot |\mathbf{b}| \sin \theta$.

   - Volume of a parallelepiped determined by the vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$:

     $$\text{Volume} = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|.$$

     Hint: Try to find the height of this parallelepiped.

Open discussion: Sep 16th & Exam 1 checklist

1. Find the following limit or explain why it does not exsit.

- $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$

  DNE. Hint: Approach $(0,0)$ along $x=0$ and $y=0$ respectively.

- $\displaystyle\lim_{(x,y)\to(1,1)} = \frac{2x^2-xy-y^2}{x^2-y^2}$

  $$\lim_{(x,y)\to(1,1)} \frac{2x^2-xy-y^2}{x^2-y^2} = \lim_{(x,y)\to(1,1)} \frac{(2x+y)(x-y)}{(x-y)(x+y)} = \lim_{(x,y)\to(1,1)} \frac{2x+y}{x+y} = \frac{3}{2}.$$

- $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+\sin y}{3x^2+y}$

  DNE. Hint: Approach $(0,0)$ along $x=0$ we have

  $$\lim_{(x,y)\to(0,0)} \frac{x^2+\sin y}{3x^2+y} = \lim_{y\to 0} \frac{\sin y}{y} = 1.$$

  Approach $(0,0)$ along $y=0$ we have

  $$\lim_{(x,y)\to(0,0)} \frac{x^2+\sin y}{3x^2+y} = \lim_{x\to 0} \frac{x^2+1}{3x^2} = \frac{1}{3}.$$

2. Determine whether the following functions are continuous at $(0,0)$ or not.

- $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y)\neq(0,0) \\ 0 & \text{if } (x,y)=(0,0) \end{cases}$

  The function is not continuous at $(0,0)$. Hint: use $y=ax$.

- $f(x,y) = \begin{cases} x\sin\dfrac{1}{y} + y\sin\dfrac{1}{x} & \text{if } xy\neq 0 \\ 0 & \text{if } xy=0 \end{cases}$

  The function is continuous at $(0,0)$:

  $$\left| x\sin\frac{1}{y} + y\sin\frac{1}{x} \right| \leq |x|+|y| \xrightarrow{(x,y)\to(0,0)} 0.$$

Exam 1 Checklist

magnitute → distance between points (scalar)

dot product (scalar) → $\cos\theta$ → angle between 2 planes/lines

$\cos\theta$ → projection → orthogonal

cross product (vector) → magnitute = area of ▱

↓

triple product (scalar) → volume of 

triple product → determine if $\vec{a}, \vec{b}, \vec{c}$ or A, B, C, D lie on the same plane

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line ⟷ directional vector $\vec{v}$ + a point on the line P

plane ⟷ normal vector $\vec{n}$ + a point on the plane P

↳ e.g. given a line l, find the plane // to l and contain a certain pt

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match level sets & figure

- check certain pts.
- symmetry (odd / even function, switch x and y)
- $f(x,y) \to \infty$ at some pt / $x, y \to \infty$.

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know the equations of qudraic surfaces

computing limits.

Open discussion: Sep 21st

1. Let $f(x,y) = e^{xy}$ and $g(x,y) = f\left(\sin(x^2+y), x^3+2y+1\right)$. Compute g_x.

   Hint: This question really tests your understanding about composing functions. If you write
   $$g(x,y) = e^{\sin(x^2+y)(x^3+2y+1)}$$
   everything should be familiar.

2. Let $f(x,y) = x^2 y$. Find the derivative of f in the direction of $\mathbf{v} = \langle 3, 4\rangle$ at the point $P = (2, \frac{1}{4})$.
   $$\nabla f(x,y) = \langle 2xy, x^2\rangle, D_{\mathbf{v}} f(P) = \langle 1, 4\rangle \cdot \frac{\langle 3,4\rangle}{|\langle 3,4\rangle|} = \frac{19}{5}.$$

3. Find the direction for which the directional derivative of $f(x,y) = 3x^2 - 4xy + 2y^2$ at $P = (\frac{1}{2}, 1)$. What is the maximum value?
   $$\nabla f(x,y) = \langle 6x - 4y, -4x + y\rangle,$$
   $$\nabla f(P) = \langle -1, 2\rangle.$$
   The maximum value of the directional derivative occurs when ∇f and the unit vector point in the same direction. Note that
   $$D_{\mathbf{v}} f = |\nabla f| \cos\theta, \theta \in [-1, 1].$$
   We can take $\cos\theta = 1$, i.e. $\theta = 0$ to maximize the directional derivative. The maximum is the magnitude $|\nabla f(P)| = \sqrt{5}$.

4. Given a differentiable function $f(u,v)$, and let $g(x,y) = f(x\cos y, \sin(x) + x^2 y)$. Using the following information to compute ∇g at $P = (\pi, 2\pi)$.

	f	g	f_u	f_v
$(\pi, 2\pi)$	1	0	4	0
$(-\pi, 4\pi^3)$	3	-2	1	2

   Chain rule:
   $$\begin{aligned}\nabla g(x,y) &= \langle g_x(x,y), g_y(x,y)\rangle \\ &= \langle f_u(u,v)\cdot u_x(x,y) + f_v(u,v)\cdot v_x(x,y), f_u(u,v)\cdot u_y(x,y) + f_v(u,v)\cdot v_y(x,y)\rangle.\end{aligned}$$
   Note that you need to use the value of f_u, f_v at the point $(u(x,y), v(x,y)) = (-\pi, 4\pi^3)$ not at the point $(\pi, 2\pi)$.
   $$\begin{aligned}\nabla g(P) &= \langle 1\cdot\cos 2\pi + 2\cdot(\cos\pi + 2\cdot\pi\cdot 2\pi), 1\cdot\pi\cdot(-\sin 2\pi) + 2\cdot\pi^2\rangle \\ &= \langle 4\pi^2 - 1, 2\pi^2\rangle.\end{aligned}$$

5. Open-ended question:

   - What is an intuitive explanation of the 1-dimensional chain rule?
     Could it be generalized to the chain rule in higher dimension?

     We only care the value of f locally when finding its derivative. In a sufficiently small neighborhood U around x, f is approximately a linear function. And notice that the chain rule works for linear functions!

     For higher dimension, chain rule is essentially matrix multiplication. It is beyond the this course but if you're interested search Jacobian matrix.

   - What is a geometric intuition for directional derivatives? Say $f : \mathbb{R}^2 \to \mathbb{R}$.

     If the domain is 1-dimensional, say $y = f(x)$ then we know $f'(x)$ gives the slope of the *tangent line*. The 2-dimensional generalization of a tangent line would be a *tangent plane* T.

     Say we are considering the directional derivative $\nabla_{\mathbf{v}} f(P)$ The extra vector $\mathbf{v}$ appear in the directional derivative gives a *slicing plane* A (the plane A is passing through the point P and parallel to the z-axis and $\mathbf{v}$). The directional derivative is given by the intersecting line of the tangent plane T and the slicing plane A.

     To see a figure, visit this clickable link.

Open discussion: Sep 23rd

1. Let P be a point in $\mathbb{R}^2$. Given that $D_{\langle 1,0\rangle} f(P) = \frac{1}{3}$ and $D_{\langle 0,-4\rangle} f(P) = 4$. Compute $D_{\langle 3,-2\rangle} f(P)$.

   Using definition of directional derivatives, we have

$$\frac{1}{3} = D_{\langle 1,0\rangle} f(P) = \nabla f(P) \cdot \langle 1, 0\rangle,$$

$$4 = D_{\langle 0,-4\rangle} f(P) = \nabla f(P) \cdot \frac{\langle 0,-4\rangle}{|\langle 0,-4\rangle|} = \nabla f(P) \cdot \langle 0, -1\rangle.$$

   Decomposing $\langle 3, -2\rangle$, we have

$$\begin{aligned} D_{\langle 3,-2\rangle} f(P) &= \nabla f \cdot \frac{\langle 3,-2\rangle}{|\langle 3,-2\rangle|} = \nabla f \cdot \frac{\langle 3,-2\rangle}{\sqrt{13}} \\ &= \nabla f \cdot \left(\frac{3}{\sqrt{13}} \langle 1, 0\rangle + \frac{2}{\sqrt{13}} \langle 0, -1\rangle \right) \\ &= \frac{3}{\sqrt{13}} \cdot \frac{1}{3} + \frac{2}{\sqrt{13}} \cdot 4 \qquad = \frac{9}{\sqrt{13}}. \end{aligned}$$

2. Suppose f is a differentiable function. Let $\mathbf{v}$ be some unit vector. At some point P, suppose $|\nabla f(P)| = 10$ and $D_{\mathbf{v}} f(P) = 5\sqrt{3}$. What is the angle between $\nabla f(P)$ and $\mathbf{v}$.

   Let's call the angel θ, then

$$\cos\theta = \frac{\nabla f(P) \cdot \mathbf{v}}{|\nabla f(P)| \cdot |\mathbf{v}|} = \frac{D_{\mathbf{v}}(P)}{|\nabla f(P)|} = \frac{\sqrt{3}}{2}.$$

3. Prove that magnitude of directional derivative $D_{\mathbf{v}} f$ is the same as component of its projection $\operatorname{proj}_{\mathbf{v}} \nabla$ on $\mathbf{v}$. That is $\operatorname{comp}_{\mathbf{v}} \nabla f = D_{\mathbf{v}} f$.

$$\operatorname{comp}_{\mathbf{v}} \nabla f = |\nabla f| \cdot \cos\theta = |\nabla f| \cdot \frac{\nabla f \cdot \mathbf{v}}{|\nabla f| \cdot |\mathbf{v}|} = \nabla f \cdot \mathbf{v} = D_{\mathbf{v}} f.$$

4. Let $z = f(x, y)$. Prove the gradient is always perpendicular to the level curves.

   Take the level curve given by $c = f(x, y)$, where c is a constant. Suppose we parametrize the level curve using the variable t, then

$$c = f(x(t), y(t)).$$

   Take derivative with respect to t on both sides, we get

$$\begin{aligned} 0 &= f_x(x(t), y(t)) \cdot x'(t) + f_y(x(t), y(t)) \cdot y'(t) \\ &= \nabla f\,(x(t), y(t)) \cdot \langle x'(t), y'(t)\rangle. \end{aligned}$$

   Note that $\langle x'(t), y'(t)\rangle$ gives the tangent direction of the level curve.

5. Use gradients and level surfaces to find the tangent plane of the graph of $z = f(x, y)$ at $P = (x_0, y_0, z_0)$.

   Consider the function
$$g(x, y, z) = f(x, y) - z.$$
   The graph of $z = f(x, y)$ is exactly the level set $g^{-1}(0) = \{(x, y, z) \mid g(x, y, z) = 0\}$. The previous question also generalize to 3-dimension (you can prove this using the same method). We know the normal vector $\mathbf{n}$ to the surface is given by
$$\nabla g = \langle g_x, g_y, g_z \rangle = \langle f_x, f_y, -1 \rangle.$$
   So the equation of the tangent plane is
$$f_x(P) \cdot (x - x_0) + f_y(P) \cdot (y - y_0) - (z - z_0) = 0.$$

Open discussion: Oct 5th & Exam 2 checklist

I took the problems from an MIT open course. Here is the detailed solution. I won't type the solution again. The plan is to go through these two problems in detail on the blackboard.

For each problem ask yourself the following question:

- What is the function f we are trying to maximize/minimize?
- What is the constraint function g? (We require $g = const.$)

Then

- Set $L = f - \lambda g$ and write down the equation $\nabla L = \nabla f - \lambda \nabla g = 0$.
- Find critical points of L and determine if it gives the maximum/minimum.

1. Find the maximum and minimum values of $f(x, y) = x^2 + x + 2y^2$ on the unit circle.

2. Find the minimum and maximum values of $f(x, y) = x^2 - xy + y^2$ on the quarter circle $x^2 + y^2 = 1$, $x, y \geq 0$.

Exam 2 Checklist

Quantative Question

compute $D_{\vec{v}}f \rightarrow$ max. min of $D_{\vec{v}}f$

compute ∇f.

tangent plane to a surface S at point P

Compute partial derivatives

Taylor expansion $\rightarrow$ linear / 2nd order approximation

find critical points $\rightarrow$ second derivative test (saddle point, local max, min)

Lagrange multiplier $\rightarrow$ critical point on a disc

Qualitative Question

Given a level curves diagram

determine the partial derivatives are $+, -, 0$

classify the critical points.

Open discussion: Oct 19th

1. Let C be the portion of the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ in the fourth quadrant, counter clockwise. Evaluate the line integral of $\mathbf{F}(x, y) = \langle x + y, 1 - x \rangle$ along the curve C.

   Parameterize the curve as $(2\cos t, 3\sin t)$, $t \in [\frac{3}{2}\pi, 2\pi]$. Then

$$\begin{aligned}
\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \int_C \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,\mathrm{d}t \\
&= \int_{\frac{3}{2}\pi}^{2\pi} \langle 2\cos t + 3\sin t, 1 - 2\cos t \rangle \cdot \langle -2\sin t, 3\cos t \rangle\,\mathrm{d}t \\
&= \int_{\frac{3}{2}\pi}^{2\pi} (-2\sin(2t) + 3\cos t - 6)\,\mathrm{d}t && = 5 - 3\pi
\end{aligned}$$

2. Find a potential function f for the vector field $\mathbf{F} = \langle 2x + 2y, 2x + 2y \rangle$.

   Hint: Set $\nabla f = \mathbf{F}$, and we obtain

$$f = x^2 + 2xy + y^2 = (x + y)^2.$$

3. Let C be the portion of the counter clockwise circle $x^2 + y^2 = 25$, from $A = (5, 0)$ to $B = (3, 4)$. Using the above f and $\mathbf{F}$, verify the Fundamental Theorem of Line Integrals, i.e.

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = f(B) - f(A).$$

   Hint: Parameterize the curve as $(5\cos t, 5\sin t)$, $t \in [0, \theta]$ where $\theta = \arctan(\frac{4}{3})$. Then

$$\begin{aligned}
RHS &= (3 + 4)^2 - (5 + 0)^2 && = 24, \\
LHS &= \int_0^\theta \langle 10\cos t + 10\sin t, 10\cos t + 10\sin t \rangle \cdot \langle -5\sin t, 5\cos t \rangle\,\mathrm{d}t \\
&= \int_0^\theta 50(\cos^2 t - \sin^2 t)\,\mathrm{d}t \\
&= \int_0^\theta 50\cos(2t)\,\mathrm{d}t = 50\sin\theta\cos\theta && = 24.
\end{aligned}$$

   The above computation used double angle formulae.

Open discussion: Oct 28th

1. Compute the following integrals

   - $\int_0^1 \int_0^1 x \max(x, y)\, \mathrm{d}A$.

     Divided the unit square into two regions by the line $y = x$. Then

     $$\begin{aligned}\int_0^1 \int_0^1 x \max(x,y)\, \mathrm{d}A &= \int_0^1 \int_0^x x \cdot x \,\mathrm{d}y\,\mathrm{d}x + \int_0^1 \int_x^1 x \cdot y\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_0^1 x^3\,\mathrm{d}x + \int_0^1 \frac{x(1-x^2)}{2}\,\mathrm{d}x \\ &= \frac{3}{8}.\end{aligned}$$

   - $\int_0^1 \int_0^{\sqrt{1-x^2}} \cos(x^2+y^2)\,\mathrm{d}y\,\mathrm{d}x$.

     Identify the integral region: R is the quarter disk in the first quadrant.

     $$\begin{aligned}\iint_R \cos(x^2+y^2)\,\mathrm{d}A &= \int_0^{\frac{\pi}{2}} \int_0^1 \cos(r^2)\, r\,\mathrm{d}r\,\mathrm{d}\theta && \text{(using polar coordinates)} \\ &= \int_0^{\frac{\pi}{2}} \frac{1}{2} \int_0^1 \cos(u)\,\mathrm{d}u\,\mathrm{d}\theta && \text{(change of variable } u = r^2\text{)} \\ &= \int_0^{\frac{\pi}{2}} \frac{1}{2}\big(\sin(1) - \sin(0)\big)\,\mathrm{d}\theta \\ &= \frac{\sin(1)\cdot\pi}{4}.\end{aligned}$$

2. Identify the integration region.

   - $\iiint_R \mathrm{d}V = \int_{-1}^1 \int_{-3\sqrt{1-x^2}}^{3\sqrt{1-x^2}} \int_{-2\sqrt{1-x^2-(y/3)^2}}^{2\sqrt{1-x^2-(y/3)^2}} \mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$

     R is an ellipsoid.

     The integral with respect to x gives $x \in [-1, 1]$. Using the upper and lower bounds for y, we can write $y^2 \le 9(1-x^2)$ i.e. $x^2 + \frac{y^2}{9} \le 1$. Similarly, $z^2 \le 4(1 - x^2 - \frac{y^2}{9})$ gives $x^2 + \frac{y^2}{9} + \frac{z^2}{4} \le 1$.

   - $\iiint_R \mathrm{d}V = \int_0^{2\pi} \int_0^1 \int_r^{2-r^2} \mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$.

     R is the region bounded by a cone and a paraboloid.

     - $z \in [r, 2-r^2]$ gives $x^2 + y^2 \le z^2$ (cone) and $z \le 2 - x^2 - y^2$ (paraboloid).
     - $r \in [0, 1]$.
     - $\theta \in [0, 2\pi]$.

Exam 3 Checklist

[4] **Line integral** $\displaystyle\int_C f(x,y)\,\mathrm{d}s = \int_a^b f\big(x(t),y(t)\big)\cdot|\mathbf{r}'(t)|\,\mathrm{d}t.$

- $f:\mathbb{R}^m\to\mathbb{R}$ is a scalar-valued function.
- $\mathbf{r}(t)$ is the parameterization of the curve C.
- $\cdot$ is scalar multiplication.

Arc length. Take $f=1$.

Vector integral $\displaystyle\int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \int_a^b \mathbf{F}\big(\mathbf{r}(t)\big)\cdot\mathbf{r}'(t)\,\mathrm{d}t$

- $\mathbf{F}:\mathbb{R}^m\to\mathbb{R}^n$ is a vector-valued function.
- $\mathbf{r}(t)$ is the parameterization of the curve C.
- $\cdot$ is dot product.

Fundamental theorem of line integral. $\displaystyle\int_C \nabla f\cdot\mathrm{d}\mathbf{r} = f\big(\mathbf{r}(b)\big)-f\big(\mathbf{r}(a)\big).$

Conservative vector field. If $\mathbf{F}=\nabla f$ for some scalar-valued potential function f, line integral only depends on endpoints.

Double integral $\displaystyle\iint_R f\,\mathrm{d}A$, where $\mathrm{d}A = \mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta.$

- R is a 2d region
- Write the integrand f as $f(x,y)$ (in rectangular case) or $f(r,\theta)$ (in polar case).
- The upper and lower bounds of your variable can be non-constant values e.g.

$$\int_{y_1}^{y_2}\int_{x_1(y)}^{x_2(y)} f(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

  However, the last integral always has constant bounds, as our final answer is a constant.

Change order of integration. Figure our the region of integration R first!!

Tripple integral $\displaystyle\iiint_R f\,\mathrm{d}V$, where $\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z = \rho^2\sin\varphi\,\mathrm{d}\rho\,\mathrm{d}\varphi\,\mathrm{d}\theta.$

- R is a 3d solid body.
- Similar comments as in the double integral.

Transformation law $\displaystyle\iint_R f(x,y)\,\mathrm{d}A = \iint_S f\big(x(u,v),y(u,v)\big)\cdot|\det J|\,\mathrm{d}A$

- J is the Jacobian matrix given by $\begin{bmatrix} x_u & x_v \\ y_u & y_v\end{bmatrix}$.
- $\cdot$ is scalar multiplication
- There also a transformation law for triple integral.

[4] Other topics required to know: how to parameterize a curve, compute potential function.

Worksheet: Nov 18

Comments on computing the flux integral in Q1.

Compute the normal vector using partial derivatives and make it a unit vector by dividing its magnitude. Note that we want the normal vector to be a unit to match the statement of the divergence theorem; and to match the physical meaning of flux.

Since the surface S_1 is a parameterized by $\mathbf{r}(u, v)$, using knowledge from Nov 8's lecture, the unit normal vector is given by

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|},$$

(Not to calculate the magnitude yet, as it will be cancelled later.)

and the infinitesimal element of surface area is given by

$$\mathrm{d}S = |\mathbf{r}_u \times \mathbf{r}_v| \,\mathrm{d}u\,\mathrm{d}v.$$

Hence,

$$\begin{aligned} \iint_{S_1} \mathbf{F} \cdot \mathbf{n} \ \mathrm{d}A &= \iint_{S_1} \mathbf{F} \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} \ |\mathbf{r}_u \times \mathbf{r}_v| \,\mathrm{d}u\,\mathrm{d}v \qquad \text{(substitute in } \mathbf{n} \text{ and } \mathrm{d}S) \\ &= \iint_{S_1} \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) \ \mathrm{d}u\,\mathrm{d}v. \end{aligned}$$

As you may have noticed, I haven't used any explicit value of the parametrization $\mathbf{r}$. So the last line is valid in general when you have a parametrization for the surface.

Exam 4 & Final Exam Checklist

[5] **Parametrization of a surface** Given a surface S parameterzied by $\mathbf{r}(u, v)$.

Normal vector. $\mathbf{r}_u \times \mathbf{r}_v$. (This may not be a unit vector.)

Infinitesimal element of surface area. $\mathrm{d}S = |\mathbf{r}_u \times \mathbf{r}_v| \,\mathrm{d}u\,\mathrm{d}v$.

Divergenece $\operatorname{div} \mathbf{F} = \dfrac{\partial F_1}{\partial x} + \dfrac{\partial F_2}{\partial y} + \dfrac{\partial F_3}{\partial z}$.

- $\mathbf{F} = \langle F_1, F_2, F_3 \rangle$.
- $\operatorname{div} \mathbf{F}$ is scalar-valued.

Curl $\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$.

- $\mathbf{F} = \langle F_1, F_2, F_3 \rangle$.
- $\operatorname{curl} \mathbf{F}$ is vector-valued.

Green's theorem $\displaystyle\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_C P\,\mathrm{d}x + Q\,\mathrm{d}y = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}S.$

- C is a curve that is positively oriented, piece smooth, simple, and closed.
- D is the region enclosed by C.
- $\mathbf{F} = \langle P, Q \rangle$ a vector field with continuous first order partial derivatives.

The divergence theorem $\displaystyle\iint_S \mathbf{F} \cdot \mathbf{n}\,\mathrm{d}S = \iiint_D \operatorname{div} \mathbf{F}\,\mathrm{d}V.$

- S is a surface that is positively oriented, piece smooth, simple, and closed.
- D is the solid region enclosed by S.
- $\mathbf{n}$ is the outward pointing, unit, normal vector to S.
- $\mathbf{F}$ a vector field with continuous first order partial derivatives.
- The LHS is called teh flux integral of $\mathbf{F}$.

Stokes' theorem $\displaystyle\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_D \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\,\mathrm{d}S.$

- C is a curve that is positively oriented, piece smooth, simple, and closed.
- D is the region enclosed by C.
- $\mathbf{n}$ is the outward pointing, unit, normal vector to C.
- $\mathbf{F}$ a vector field with continuous first order partial derivatives.

[5] Other topics required to know: how to parameterize a surface, compute divergence and curl of a vector field.

Changing order of integration without drawing

Let's take the following integral as an example.

$$\int_0^9 \int_0^{x^2} \int_0^y f(x,y,z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x.$$

Now the integrating order is z, y, x. Using the integral, we obtain the following inequalities:

$$\begin{aligned}
&\int_0^y \cdots \mathrm{d}z && \implies 0 \le z \le y && (A)\\
&\int_0^{x^2} \cdots \mathrm{d}y && \implies 0 \le y \le x^2 && (B)\\
&\int_0^y \cdots \mathrm{d}z && \implies 0 \le x \le 9 && (C)
\end{aligned}$$

Let's change the order of integration to be x, y, z:

Lower bound of x. Inequality, (B) gives $\sqrt{y} \le x$; inequality (C) gives $0 \le x$. Since $0 \le \sqrt{y}$, the lower bound of x is $\sqrt{y}$. [6]

Upper bound of x. Inequality (C) gives $x \le 9$. So 9 is the upper bound.

$$\implies \int_{\sqrt{y}}^9 \cdots \mathrm{d}x.$$

Lower bound of y. Inequality (A) gives $z \le y$; inequality (B) gives $0 \le y$. Since $0 \le z$, the lower bound of y is z.

Upper bound of y. Inequality (B) gives $y \le x^2$. [7] Substituting the upper bound of x gives $y \le x^2 = 81$. So 81 is the upper bound.

$$\implies \int_z^{81} \cdots \mathrm{d}y.$$

Lower bound of z. Inequality (A) gives $0 \le z$.

Upper bound of z. Inequality (A) gives $z \le y$. Substituting the upper bound of y gives $z \le y = 81$.

$$\implies \int_0^{81} \cdots \mathrm{d}z.$$

Hence [8]

$$\int_0^9 \int_0^{x^2} \int_0^y f(x,y,z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x = \int_0^{81} \int_z^{81} \int_{\sqrt{y}}^9 f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

[6] Intuitively once the variable reaches $\sqrt{y}$, it may not descent further.

[7] When we compute the definite integral with respect to x, the answer does not depend on x, since we substitute in the upper and lower bounds.

[8] Check it on a triple integral calculator if you don't believe it.